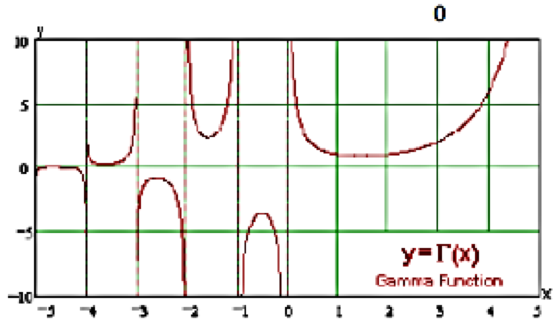
Gamma and Beta Funtions دوال كاما وبيتا

دالة كاما : تعرف دالة كاما كالآتي :

$$\Gamma(n)=\lim_{M\to\infty}\int_0^M x^{n-1}e^{-x}dx \quad ; n>0$$



فمثلاً لإيجاد $\Gamma(2)$

$$\Gamma(2)=\lim_{M\to\infty}\int_0^M x^{2-1}e^{-x}dx=\lim_{M\to\infty}\int_0^M x\ \ e^{-x}dx$$

x & $D.$		e^{-x} & $I.$
x	+	e^{-x}
1	-	$-e^{-x}$
0		e^{-x}

$$=\lim_{M\to\infty}[-xe^{-x}-e^{-x}|_0^M=\lim_{M\to\infty}\left(\frac{-M}{e^M}-\frac{1}{e^M}+0+e^0\right)=1$$

قواعد اساسية

1. $\Gamma(n+1)=n\Gamma(n) \qquad \forall\, n\neq 0$

2. $\Gamma(n+1)=n!$ واذا كانت n عدداً صحيحاً موجباً فان

3. $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

ملاحظة : لا يمكن ايجاد $\Gamma(n)$ اذا كان n عدداً صحيحاً سالباً .

(أ)

خصائص دالة كاما

① $\Gamma(1) = 1$

<u>Proof:</u>

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = \lim_{b \to \infty} \int_0^b e^{-t}\, dt$$

$$= \lim_{b \to \infty} \left[-e^{-t} \right]_0^b$$

$$= \lim_{b \to \infty} \left[-e^{-b} + e^{0} \right]$$

$$= 1$$

ملاحظة

$e^{-\infty} \xrightarrow{\text{تقترب}} 0$

$\frac{1}{e^{\infty}} = \frac{1}{\infty} \Rightarrow 0$

(2) $\Gamma(x+1) = x\Gamma x$

Proof

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$$

$$\Gamma(x+1) = \int_0^\infty e^{-t}\, t^{x\overbrace{+1-1}^{=0}}\, dt$$

using integration by parts

$$u = t^x \Rightarrow du = x\, t^{x-1}\, dt$$

$$dv = e^{-t} dt \Rightarrow v = -e^{-t}$$

$$\therefore \Gamma(x+1) = \left[-e^{-t}\, t^x\right]_0^\infty - \int_0^\infty -e^{-t} \cdot x t^{x-1}\, dt$$

$$= (-e^{-\infty}\, \infty^x) - (-e^{-0}\, 0^x) - \int_0^\infty e^{-t} \cdot x t^{x-1}\, dt$$

$$= \left(\frac{-1}{e^{\infty}}\, \infty^x\right) - (-e^{0}\, 0^x) - x\int_0^\infty e^{-t}\, t^{x-1}\, dt$$

$$= 0 + x \boxed{\int_0^\infty e^{-t}\, t^{x-1}\, dt} = x\Gamma x$$

$$\therefore \Gamma(x+1) = x\Gamma x$$

③ $\Gamma(x+1) = x!$; x is integer & possitive

Proof $\Gamma(x)+1) = x!$

$$\Gamma(x+1) = x\,\Gamma(x)$$
$$= x\,(x-1)\,\Gamma(x-1)$$
$$= x\,(x-1)\,(x-2)\,\Gamma(x-2)$$
$$\vdots$$
$$= x\,(x-1)\,(x-2)\cdots 3\cdot 2\cdot 1\,\Gamma(1) = x!$$

مثال (١) : احسب قيمة

1. $\Gamma(4)$ 2. $\dfrac{\Gamma(6)}{2\Gamma(3)}$ 3. $\dfrac{\Gamma(3/2)}{\Gamma(1/2)}$

4. 5. $\Gamma\left(\dfrac{5}{2}\right)$ 6. $\dfrac{\Gamma(8/3)}{\Gamma(2/3)}$

الحل :

1. $\Gamma(4) = 3! = 3 \times 2 \times 1 = 6$

2. $$\frac{\Gamma(6)}{2\Gamma(3)} = \frac{5!}{2 \times 2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 1} = 30$$

3. $$\frac{\Gamma(3/2)}{\Gamma(1/2)} = \frac{\Gamma\big(1 + (1/2)\big)}{\Gamma(1/2)} = \frac{\frac{1}{2}\Gamma(1/2)}{\Gamma(1/2)} = \frac{1}{2}$$

4.

5. $$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(1 + \frac{3}{2}\right) = \frac{3}{2} \times \Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \times \Gamma\left(1 + \frac{1}{2}\right)$$

$$= \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{4}$$

6. $$\frac{\Gamma\left(\frac{8}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\Gamma\left(1 + \frac{5}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\frac{5}{3}\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{\frac{5}{3}\Gamma\left(1 + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}$$

$$= \frac{\frac{5}{3} \times \frac{2}{3}\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \frac{10}{9}$$