دالة بيتا : تعرف دالة بيتا كالآتي :

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad ; \; n>0\,, \quad m>0$$

فمثلاً لإيجاد $B(2,3)$

$$B(2,3)=\int_0^1 x\;(1-x)^2\,dx$$

$$=\int_0^1 x(1-2x+x^2)dx$$

$$=\int_0^1 (x-2x^2+x^3)dx=\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}\Big|_0^1=\frac{1}{2}-\frac{2}{3}+\frac{1}{4}=\frac{1}{12}$$

ملاحظة: $B(m,n)=B(n,m)$

علاقة دالة بيتا بدالة كاما

$$\boxed{B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

فمثلاً لإيجاد $B(2,3)$

$$B(2,3)=\frac{\Gamma(2)\Gamma(3)}{\Gamma(2+3)}=\frac{1!\,2!}{4!}=\frac{1\times 2\times 1}{4\times 3\times 2\times 1}=\frac{1}{12}$$

تكاملات كثيرة يمكن ايجادها باستعمال الدالتين بيتا و كاما منها :

1. $$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\frac{1}{2}B(m,n)$$

2. $$\int_0^{\infty}\frac{x^{p-1}}{1+x}dx=\Gamma(p)\Gamma(1-p)=\frac{\pi}{\sin p\pi} \quad ; \quad 0<p<1$$

* **Beta Function:**

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

Find $\beta(1,1)$

$$\beta(1,1) = \int_0^1 dt = t\Big]_0^1 = \boxed{1}$$

* اهم خاصيه من خصائص داله بيتا :

$$\beta(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

(1) $\beta(x,y) = \beta(y,x)$

Proof:

$$\beta(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \frac{\Gamma(y)\,\Gamma(x)}{\Gamma(y+x)} = \beta(y,x)$$

1. $\beta(x,1) = \frac{1}{x}$

$$\beta(x,1) = \frac{\Gamma(x)\Gamma(1)}{\Gamma(x+1)} = \frac{\cancel{\Gamma x}}{x\cancel{\Gamma(x)}} = \frac{1}{x}$$

2. $\beta(1,y) = \frac{1}{y}$

$$\beta(1,y) = \frac{\Gamma(1)\Gamma(y)}{\Gamma(1+y)} = \frac{\cancel{\Gamma(y)}}{y\cancel{\Gamma(y)}} = \frac{1}{y}$$

3. $\beta(1,1-x) = \frac{1}{1-x}$

$$\beta(1,1-x) = \frac{\Gamma(1)\Gamma(1-x)}{\Gamma(1+1-x)} = \frac{\Gamma(1)\Gamma(1-x)}{\Gamma(2-x)}$$

$$= \frac{\cancel{\Gamma(1-x)}}{1-x\,\cancel{\Gamma(1-x)}} = \frac{1}{1-x}$$

4. $\beta(x,x+1) = \frac{(\Gamma(x))^2}{2\Gamma(2x)}$

$$\beta(x,x+1) = \frac{\Gamma(x)\Gamma(x+1)}{\Gamma(2x+1)} = \frac{\Gamma(x)\,\cancel{x}\Gamma(x)}{2\cancel{x}\,\Gamma(2x)}$$

$$= \frac{(\Gamma(x))^2}{2\Gamma(2x)}$$

(5) $\beta\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\pi}{2}$

$$\beta\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\underbrace{\Gamma(2)}_{1}}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}\,\sqrt{\pi} = \frac{\pi}{2}$$

(6) $\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$ H.W.

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\underbrace{\Gamma(1)}_{1}} = \frac{\sqrt{\pi}\,\sqrt{\pi}}{1} = \pi$$