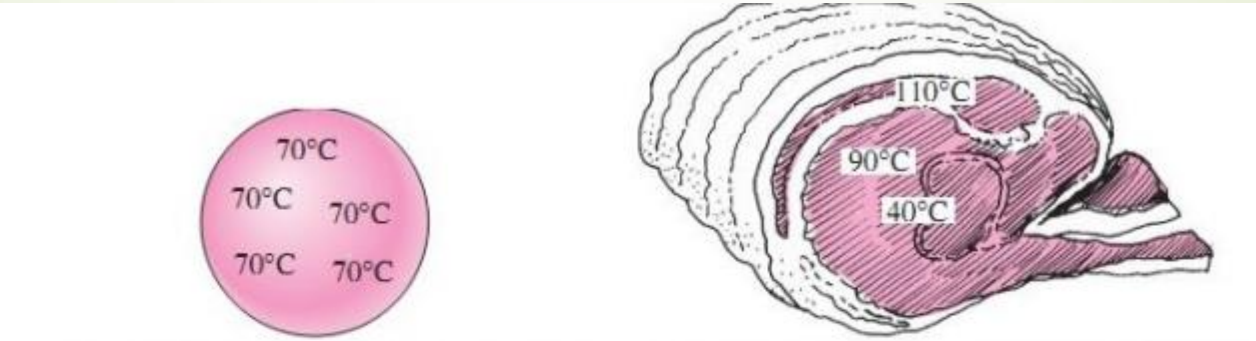
Transient Heat conduction

- In this case the temperature be function on time as it is function of dimensions.
- $T = T(x, y, z, \tau)$
- **LUMPED SYSTEM CONCEPT**

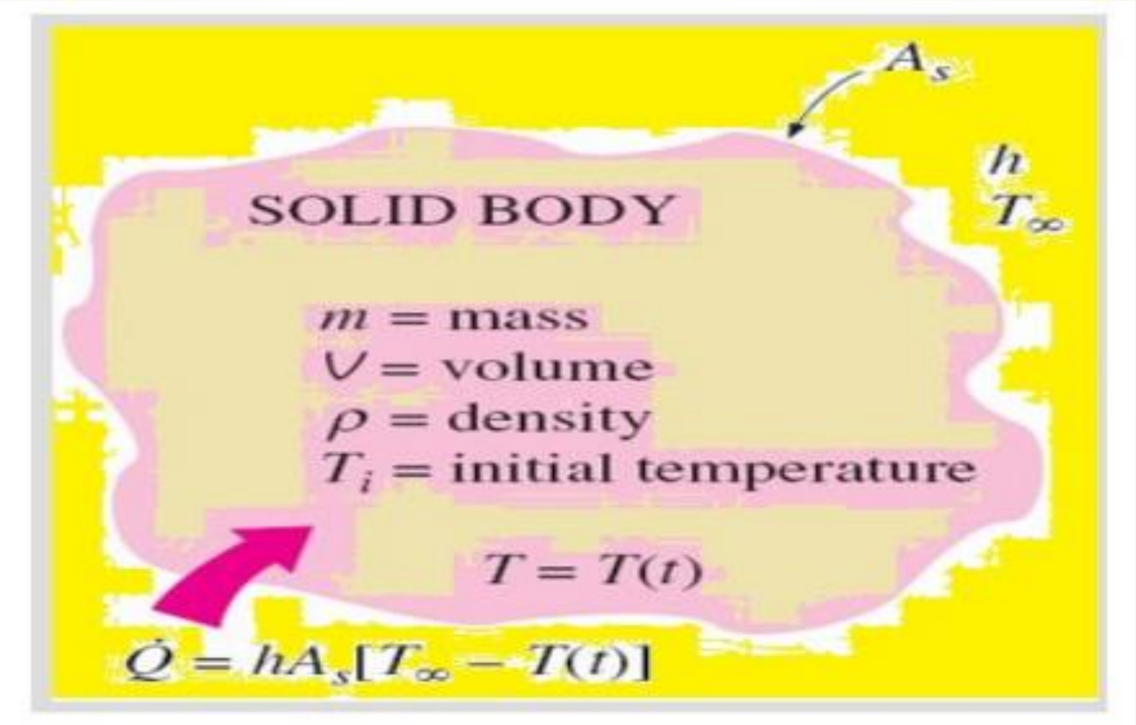


(a) metal with constant Temp. distribution

(b) Roast beef with variation in temp. with location

- Consider a cold solid of arbitrary shape, with mass m, initially at a uniform temperature To, suddenly immersed into a higher-temperature environment. As heat flows the hot environment into the cold body, the temperature of the solid increases. It is assumed that the lumped system approximation is applicable, namely, that the distribution of temperature within the solid at any instant can be regarded as almost uniform (i.e., the temperature gradients within the solid are neglected).

$$\left\{\begin{matrix}\text{Increase of the Internal}\\ \text{Energy of the Solid Over}\\ \text{the Time Interval } d\tau\end{matrix}\right\} = \left\{\begin{matrix}\text{Heat Transfer to the Solid}\\ \text{Through the outer surface}\\ \text{Over the Time Interval } d\tau\end{matrix}\right\}$$



$$\left\{\begin{matrix} Increase\ of\ the\ Internal \\ Energy\ of\ the\ Solid\ Over \\ the\ Time\ Interval\ d\tau \end{matrix}\right\} = (mass)CdT = \rho VCdT$$

Where ρ, C, and V are density, specific heat, and volume of the solid body respectively. The temperature of the solid body T is a function of time, $T=T(\tau)$.

Let $Q(\tau)$ be the total heat rate following into the body through its boundary surfaces at any instant τ.

$$\left\{\begin{matrix} Heat\ Transfer\ to\ the\ Solid \\ Through\ the\ Outer\ Surface \\ Over\ the\ Time\ Interval\ d\tau \end{matrix}\right\} = \dot{Q}(\tau)d\tau$$

$$\rho V C dT = \dot{Q}(\tau) d\tau$$

$$\frac{dT(\tau)}{d\tau} = \frac{\dot{Q}(\tau)}{\rho V C}$$

For Convection Heat Transfer

$$\dot{Q}(\tau) = A_s h[T_\infty - T(\tau)]$$

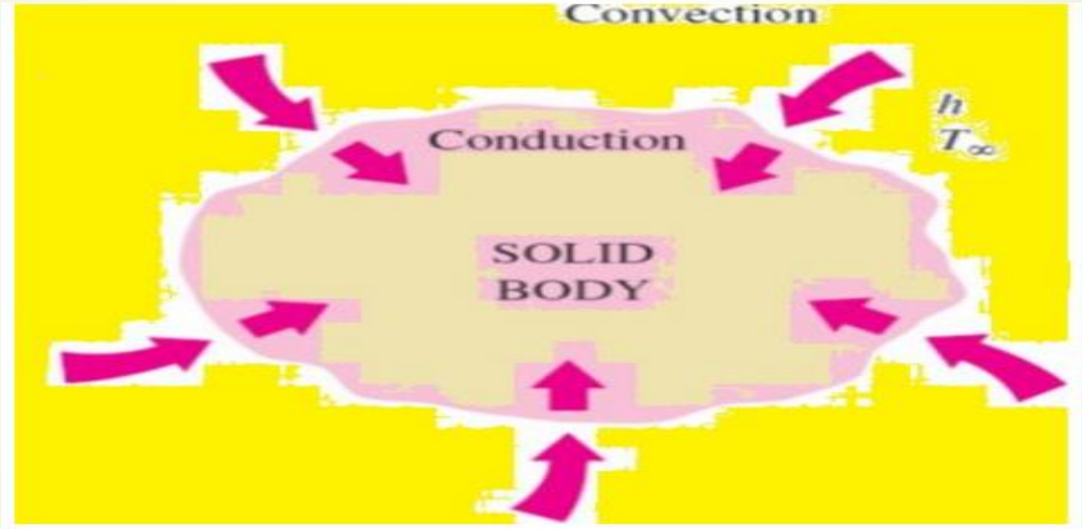
$$\frac{dT(\tau)}{d\tau} = \frac{A_s h}{\rho V C}[T_\infty - T(\tau)]$$

$$\frac{dT(\tau)}{d\tau} = -\frac{A_s h}{\rho V C}[T(\tau) - T_\infty] \qquad \text{for } \tau > 0$$

$$T(\tau) = T_i \qquad \text{for } \tau = 0$$

- For convenience in the analysis, we measure the temperature in excess of the ambient temperature T_∞; that is, we choose T_∞ as the reference temperature. Then assume that:

$$\theta(\tau) = T(\tau) - T_\infty$$



$$\theta_i = T_i - T_\infty \quad \& \quad \frac{dT(\tau)}{d\tau} = \frac{d\theta(\tau)}{d\tau}$$

And also a quantity m is introduced as $m = \dfrac{A_s h}{\rho V C}$

- Where m has the dimension of $(time)^{-1}$.

$$\frac{d\theta(\tau)}{d\tau} + m\theta = 0 \qquad \text{for } \tau > 0$$

$$\theta(\tau) = \theta_i \qquad \text{for } \tau = 0$$

And by the separating of variables, we get

$$\frac{d\theta(\tau)}{\theta(\tau)} = -m\,d\tau$$

The integration of this equation will give

$$\ln\theta(\tau) = -m\tau + C$$

Where C is the integrating constant.

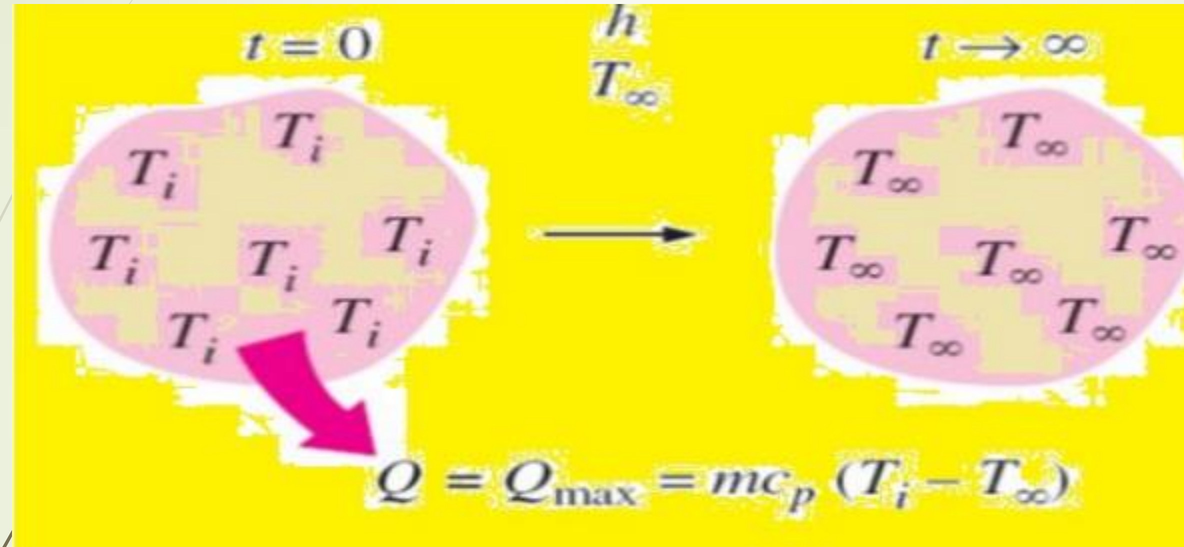
$$\ln\theta_i = -(m)0 + C \rightarrow \quad C = \ln\theta_i$$

$$\ln\theta(\tau) = -m\tau + \ln\theta_i$$

$$\ln\theta(\tau) - \ln\theta_i = -m\tau \rightarrow \qquad \ln\left(\frac{\theta(\tau)}{\theta_i}\right) = -m\tau$$

$$\frac{\theta(\tau)}{\theta_i} = e^{-m\tau}$$

$$\frac{T(\tau)-T_\infty}{T_i-T_\infty}=e^{-\frac{A_s h}{\rho V C}\tau}$$



$$L_c=\frac{V}{A_s}$$

$$m=\frac{h}{\rho C L_c}$$

We can now find the relation of the characteristic length (L_c) value for some geometries such as

- $L_C = \frac{V}{A_S}$

1- A solid sphere $L_c = \frac{\frac{4}{3}\pi R^3}{4\pi R^2} = \frac{R}{3} = \frac{D}{6}$

2- A solid cube $L_c = \frac{L^3}{6L^2} = \frac{L}{6}$

3- A solid cylinder of length L $L_c = \frac{\pi L R^2}{2\pi R L + 2\pi R^2} = \frac{LR}{2(L+R)}$

4- A solid long cylinder $L_c = \frac{\pi R^2 L}{2\pi R L} = \frac{R}{2} = \frac{D}{4}$

5- A solid large plate with thickness L and one side surface area A*

$$L_c = \frac{A^* L}{2A^*} = \frac{L}{2}$$

Criteria For Lumped system Analysis

$$Bi = \frac{Convection\ at\ The\ Surface\ of\ the\ Body}{Conduction\ Within\ The\ Body} = \frac{h}{k/L_c}\frac{\Delta T}{\Delta T} = \frac{hL_c}{k}$$

$$Bi = \frac{Conduction\ Resistance\ Within\ the\ Body}{Convection\ Resistance\ at\ The\ Surface\ of\ the\ Body} = \frac{L_c/k}{1/h} = \frac{hL_c}{k}$$

A small Biot number means that a small heat conduction resistance and thus a small temperature gradient within the body. In general, the analysis lumped system can be applied

$$Bi \leq 0.1$$

The parameter ($m\tau$) can be modified by the following

$$m\tau = \frac{h}{\rho C L_c}\tau = \frac{h}{\rho C L_c}\frac{kL_c}{kL_c}\tau = \frac{hL_c}{k}\frac{k}{\rho C}\frac{1}{L_c^2}\tau$$

$$m\tau = \frac{hL_c}{k}\frac{\alpha\tau}{L_c^2} = Bi \cdot Fo$$

Where **Fo** is the **Fourier Number**

$$\frac{T(\tau) - T_\infty}{T_i - T_\infty} = e^{-Bi \cdot Fo}$$

- **Example** **Using lumped system method, determine the time required for a solid steel ball of radius (3cm), thermal conductivity (55W/m.^{o}C), density (7830kg/m^3), and specific heat (460J/kg.^{o}C) to cool from (1000^oC) to (250^oC). If the ball is exposed to stream of air at (100^oC)having a coefficient of heat transfer (100W/$m^2.^o C$).**
- **Solution**: In this problem the time of cooling steel ball from (1000^oC) to (250^oC) by using the lumped capacity method when the ball is facing to convection heat transfer with coefficient of heat transfer of (100W/$m^2.^o C$) and environment temperature of (100^oC). The radius of the ball R=0.03m.

- **Assumption:** The ball material thermal properties and the coefficient of heat transfer are constant. The radiation effect is negligible. There is no temperature gradient through the ball.
- **Properties:** The properties of the ball material are constant and they are k=55W/m.oC, ρ=$7830 kg/m^3$, C=460J/$kg.^o C$.
- **Analysis:** The characteristic length of the ball with R=3cm is:
- $L_c = \frac{R}{3} = \frac{0.03}{3} = 0.01m$
- $m = \frac{h}{\rho C L_c} = \frac{100}{7830 \times 460 \times 0.01} = 0.00278 s^{-1}$
- Now to find the time spending for cooling is

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$$\frac{T(\tau)-T_\infty}{T_i-T_\infty} = e^{-m\tau} \qquad \rightarrow \frac{250-100}{1000-100} = -0.00277\tau$$

- $\tau = -\frac{1}{0.00277} ln\left(\frac{150}{900}\right) = 646.8sec = 10.78min$
- The checking for lumped system criteria we can find Biot number
- $Bi = \frac{hL_c}{K} = \frac{100\times0.01}{55} = 0.018 > 0.1$