Heat Transfer by Extended Surfaces (Fins)

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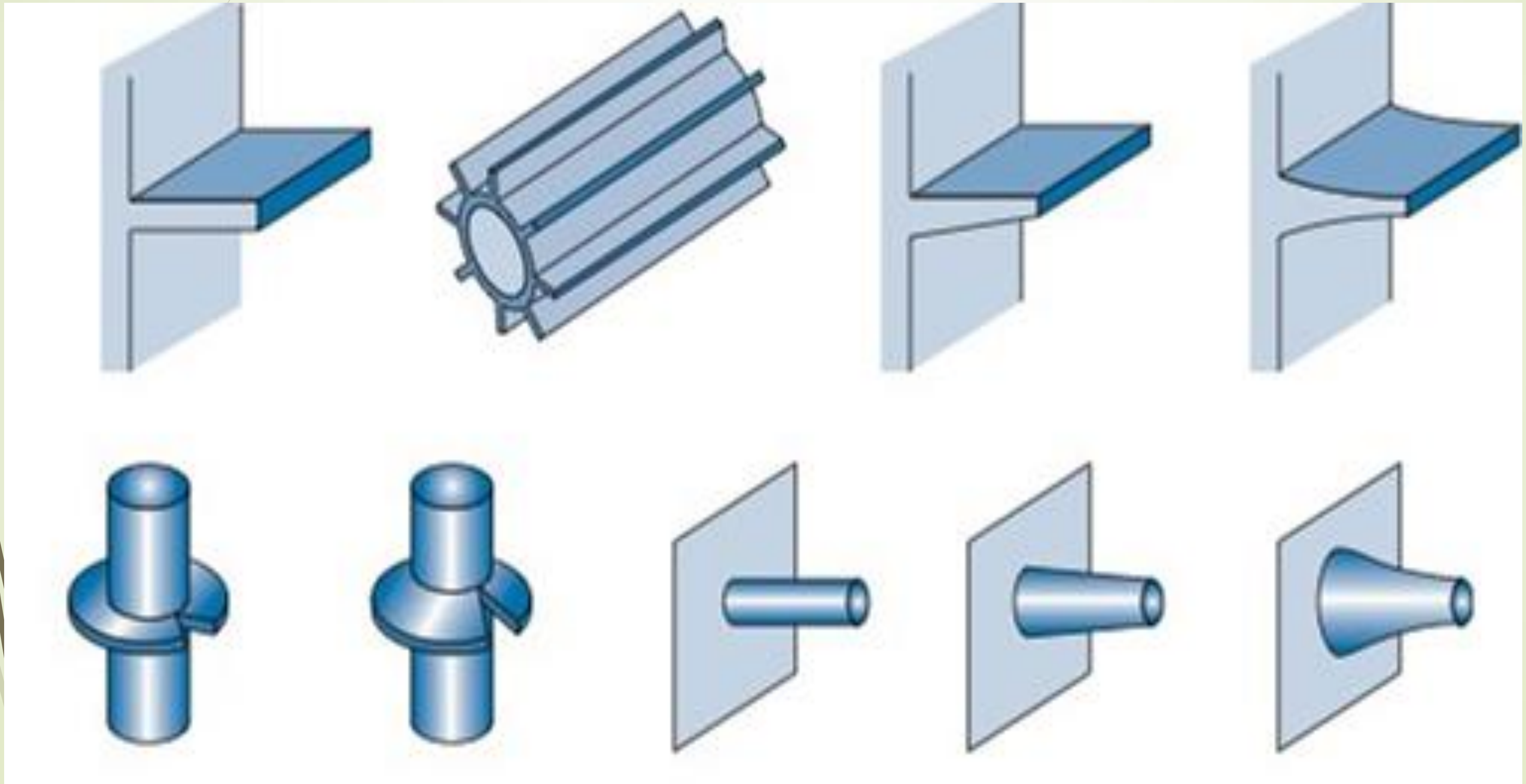
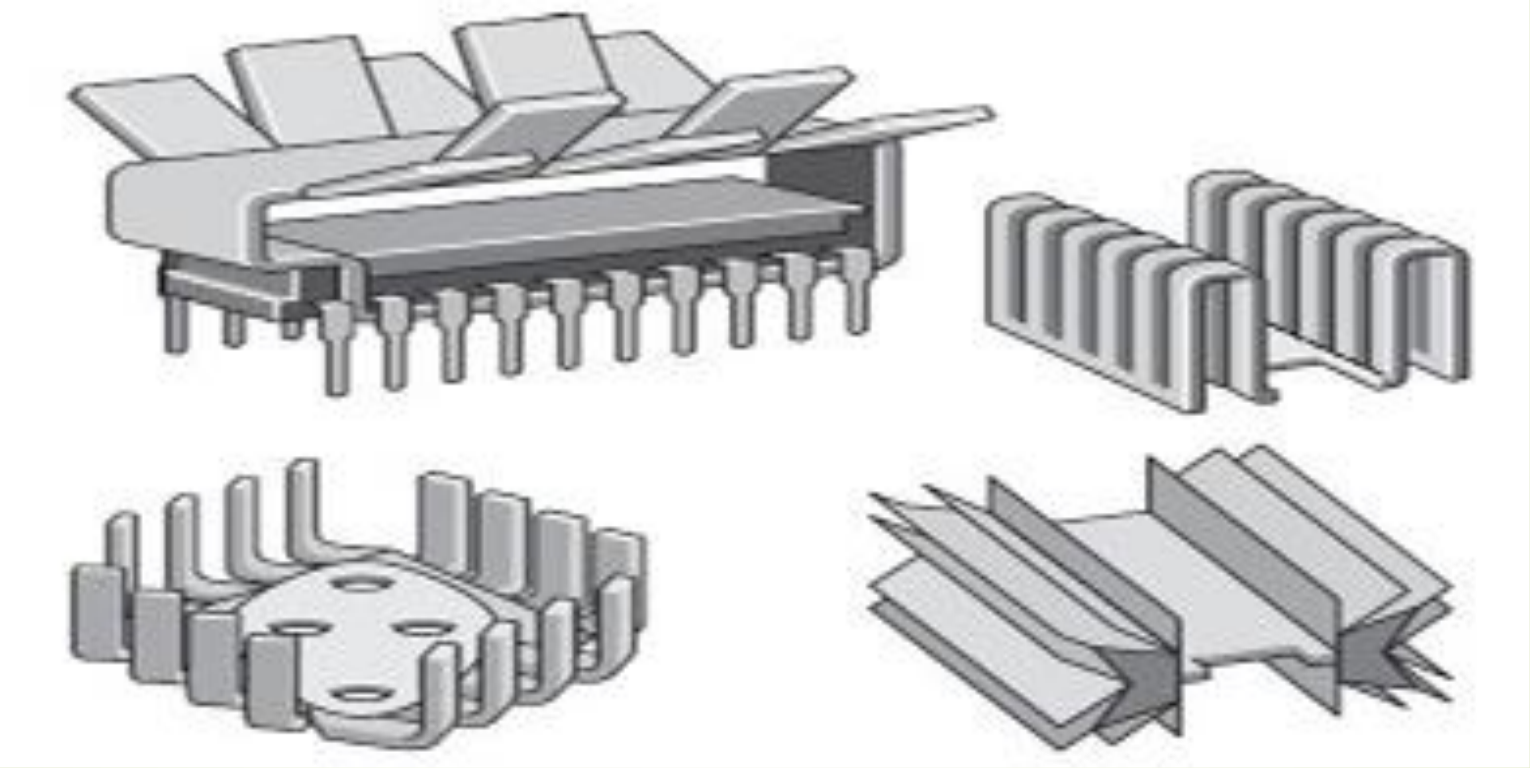
Heat transfer from Extended Surfaces

- The heat transfer by convection is increased by increasing the area that exposed to convection. This led to increase the area of heat transfer in some application that need high heat transfer rate. The increasing of the area of heat transfer is done by extending the surfaces that exposed to convection. This extended surfaces is called fines as shown in Figure

Fins

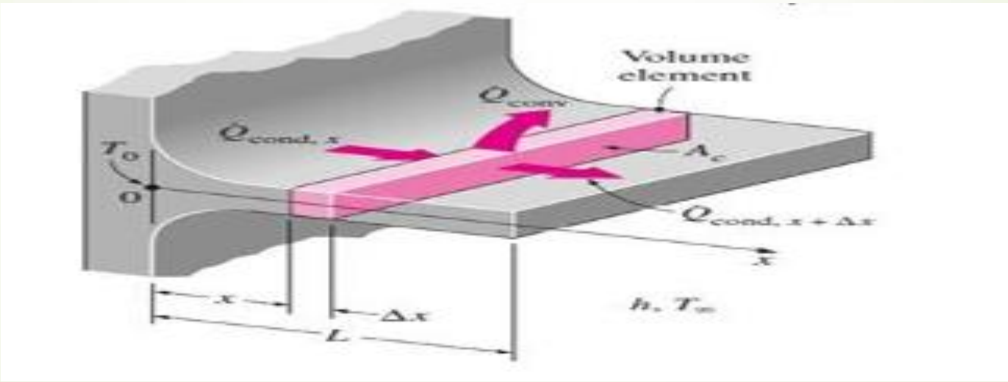


Fins



Equation of Fin

- Let us consider an element from a fin at x location with length Δx and sectional area A_c and its perimeter of p as shown in Figure. The equation of energy balance for steady state condition of this element can be expressed as:



$$\begin{Bmatrix} Conduction\ Heat \\ Transfer\ Rate\ into \\ The\ Element\ at\ x \end{Bmatrix} - \begin{Bmatrix} Conduction\ Heat \\ Transfer\ Rate\ from \\ The\ Element\ at\ x+\Delta x \end{Bmatrix} = \begin{Bmatrix} Convection\ Heat \\ Transfer\ Rate\ from \\ The\ Element \end{Bmatrix}$$

$$\dot{Q}_{cond,x} - \dot{Q}_{cond,x+\Delta x} = \dot{Q}_{conv}$$

And $\dot{Q}_{cond.x+dx} = -kA\frac{dT}{dx} + \frac{d}{dx}\left(-kA\frac{dT}{dx}\right)dx$

And $\dot{Q}_{conv} = hPdx(T - T_\infty)$

$$-kA\frac{dT}{dx} - \left[-kA\frac{dT}{dx} + \frac{d}{dx}\left(-kA\frac{dT}{dx}\right)dx\right] = hPdx(T - T_\infty)$$

- $kA\frac{d^2T}{dx^2}dx = hpdx(T - T_\infty)$
- By dividing this equation by kAdx, we get
- $\frac{d^2T}{dx^2} - \frac{hp}{kA}(T - T_\infty) = 0$
- By assuming $\theta = T - T_\infty$ $then$ $\frac{d\theta}{dx} = \frac{dT}{dx}$ and
- $\frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2}$. And $\frac{hP}{kA} = m^2$
- $\frac{d^2\theta}{dx^2} - m^2\theta = 0$
- The solution of this equation is

- $\theta = C_1 e^{-mx} + C_2 e^{mx}$ (A)

- Where C1 and C2 are integration constants of arbitrarily whose values are to be determined by the applying of the boundary conditions.
- These boundary conditions are at the base and at the tip of the fin.
- The only two conditions are to be needed to determine C1 and C2 which are uniquely.
- B.C.1 is that temperature at the base of fin is known at its value is $at\ x = 0\ \ T = T_o$
- Then $\theta = T_o - T_\infty \rightarrow \ \theta = \theta_o$

- By substituting in eq.(A) x=0 and $\theta = \theta_o$

$$\theta_o = C_1 + C_2 \qquad (1)$$

- The second boundary depending on the free end of the fin. There are three cases for this end.
- Case 1. the fin is very long that at x=∞ $T = T_\infty$
- It means that at x=∞ $\theta = T_\infty - T_\infty = 0$
- Then $\theta = C_1 e^{-m\infty} + C_2 e^{m\infty} = 0 \qquad (2)$
- From this equation $C_1 \neq 0,$ then $C_2 = 0$
- By substituting this in eq.(1) we get
- $C_1 = \theta_o$

- $\theta = \theta_o e^{-mx}$ (T.D.E) for long fin
- $$\frac{\theta}{\theta_o} = \frac{T - T_\infty}{T_o - T_\infty} = e^{-\sqrt{\frac{hp}{kA}}x} \qquad (3)$$

1- long fin

- Heat transfer from fin is equal to heat flow from its base by conduction
- $$\dot{Q}_{fin} = -kA\frac{dT}{dx})_{x=0} = -kA\frac{d\theta}{dx})_{x=0} = -kA\frac{d}{dx}\left(\theta_o e^{-\sqrt{\frac{hP}{kA}}x}\right)_{x=0}$$
  $$= -kA\left(-\theta_o\sqrt{\frac{hP}{kA}}\right)e^{-0} = \sqrt{hPkA}\,\theta_o$$
- $$\dot{Q}_{fin} = \sqrt{hPkA}(T_o - T_\infty)$$

- Also heat transfer from fin is equal to heat transfer by convection
- from all the fin.
- $\dot{Q}_{fin} = \int_o^{\infty} hP(T - T_{\infty})dx = \int_0^{\infty} hP\theta dx$
- $\dot{Q}_{fin} = hP \int_0^{\infty} \theta_o e^{-mx} dx = -hP \frac{\theta_o}{m} e^{-mx}]_0^{\infty}$
- $\dot{Q}_{fin} = -\frac{hP\theta_o}{\sqrt{\frac{hP}{kA}}}(e^{-\infty} - e^0) = -\sqrt{hPkA}\theta_o(0 - 1)$
- $\dot{Q}_{fin} = \sqrt{hPkA}\theta_o$
- It equal to that transfer from the base so it is O.K.

2- short Insulated end fin

in this case the fin is of known length and its end is insulated.

The boundary conditions are

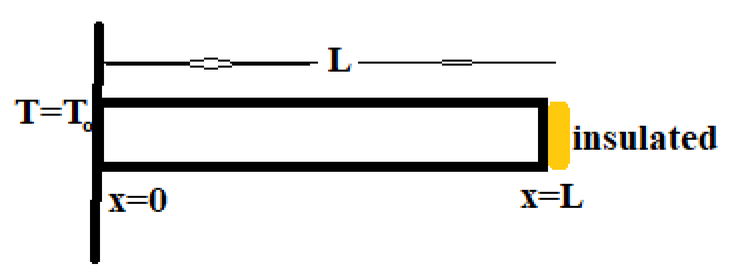
B.C.1 is as in eq.(1) $T = T_o$ at x=0

B.C.2 is at x=L $\frac{dT}{dx} = 0$ for insulated end.

The T.D.E is $\frac{\theta(x)}{\theta_o} = \frac{T(x) - T_\infty}{T_o - T_\infty} = \frac{\cosh[m(L-x)]}{\cosh(mL)}$

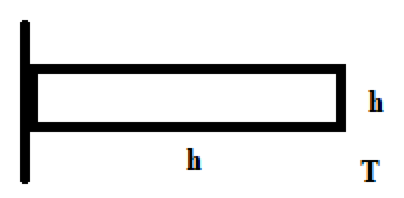
Heat transfer from the fin is

$$\dot{Q}_{fin} = \sqrt{hPkA}\,\theta_o \tanh(mL)$$



3- Short non insulated end fin

Fin with heat transfer by convection from its end.



B.C.1 is as in eq.1 at x=0 $T = T_o$ or $\theta = \theta_o$

B.C.2 at x=L $-k\frac{d\theta}{dx})_{x=L} = h\theta_L$

$$\frac{\theta(x)}{\theta_o} = \frac{T(x) - T_\infty}{T_o - T_\infty} = \frac{\cosh[m(L-x)] + \frac{h}{mk}\sinh[m(L-x)]}{\cosh(mL) + \frac{h}{mk}\sinh(mL)}$$

$$\dot{Q}_{Fin} = \sqrt{hpkA_c}\,\theta_o \frac{\sinh(mL) + \frac{h}{km}\cosh(mL)}{\cosh(mL) + \frac{h}{km}\sinh(mL)}$$

- **Fin efficiency**

$$\eta_{Fin} = \frac{[Actual\ Heat\ Transfer\ from\ Fin]}{\left[\begin{array}{l} Heat\ Transfer\ from\ Fin\ When \\ The\ Entire\ Fin\ Area\ at\ Base\ Temp. \end{array}\right]}$$

$$\eta_{Fin} = \frac{\dot{Q}_{Fin}}{hA_{fin}\theta_o}$$

The efficiency of long fin is

$$\eta_{Fin} = \frac{\dot{Q}_{Fin}}{hA_{fin}\theta_o} = \frac{\sqrt{hpkA_c}\,\theta_o}{hpL\theta_o} = \sqrt{\frac{kA_c}{hp}}\frac{1}{L} = \frac{1}{mL}$$

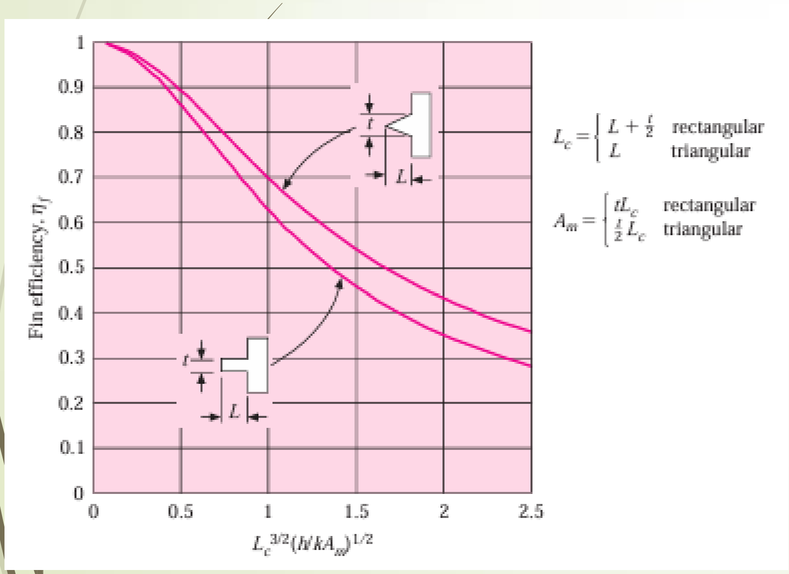
Fin effectiveness

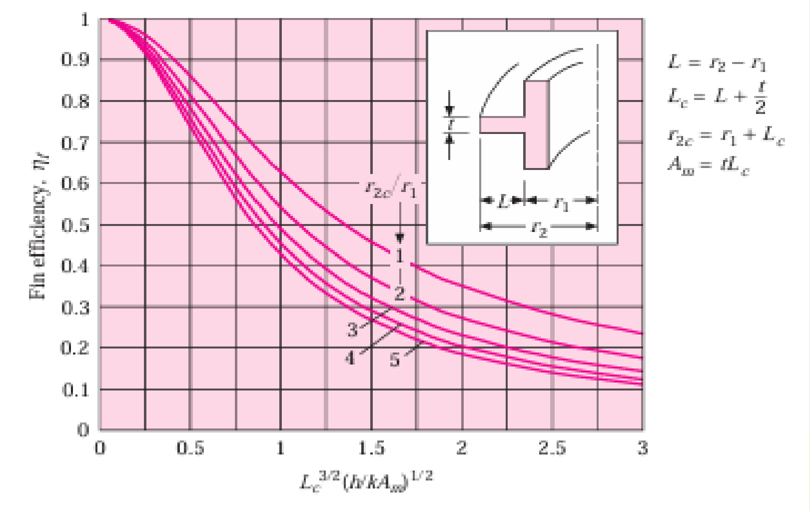
$$\varepsilon_{fin} = \frac{\left[\begin{array}{l} Heat\ Transfer\ Rate\ From \\ The\ Fin\ of\ Base\ Area\ A_o \end{array}\right]}{\left[\begin{array}{l} Heat\ Transfer\ Rate\ From \\ The\ Surface\ of\ Area\ A_o \end{array}\right]} = \frac{\dot{Q}_{Fin}}{\dot{Q}_{no\ Fin}} = \frac{\dot{Q}_{Fin}}{A_o h(T_o - T_\infty)}$$

- And also we can find the flowing charts to find the efficiency of fin.

- heat transfer from the fin

- $$\dot{Q}_f = \eta_f A_f h (T_o - T_\infty)$$





Long Fin specifications

Specifications	Mathematic Relation
Boundary conditions	B.C.1 x=0 $T = T_o$ or $\theta = \theta_o = (T_o - T_\infty)$ B.C.2 x=∞ $T = T_\infty$ or $\theta = 0 = (T_\infty - T_\infty) = 0$
T.D.E	$\theta = \theta_o e^{-mx}$ or $\frac{T-T_\infty}{T_o-T_\infty} = exp\left(-\sqrt{\frac{hP}{kA}}x\right)$
Heat transfer	$\dot{Q}_{fin} = \sqrt{hPkA}\theta_o = \sqrt{hPkA}(T_o - T_\infty)$
Fin efficiency η_f	$\eta_{fin} = \frac{1}{mL} = \sqrt{\frac{kA}{hP}\frac{1}{L}}$
Fin effectiveness ε_f	$\varepsilon_{fin} = \sqrt{\frac{Pk}{A_o h}}$

Insulated end fin

Specifications	Mathematic Relation
Boundary conditions	B.C.1 x=0 $T = T_o$ or $\theta = \theta_o = (T_o - T_\infty)$ B.C.2 x=L $\frac{dT}{dx} = 0$ or $\frac{d\theta}{dx} = 0$
T.D.E	$\frac{\theta}{\theta_o} = \frac{T - T_\infty}{T_o - T_\infty} = \frac{\cosh[m(L-x)]}{\cosh(mL)}$
Heat transfer	$\dot{Q}_{fin} = \sqrt{hPkA}\,\theta_o \tanh(mL)$ $\dot{Q}_{fin} = \sqrt{hPkA}(T_o - T_\infty)\tanh(mL)$
Fin efficiency η_f	$\eta_{fin} = \frac{\tanh(mL)}{mL}$
Fin effectiveness ε_f	$\varepsilon_{fin} = \sqrt{\frac{Pk}{A_o h}}\tanh(mL)$

Convection from end tip

Specifications	Mathematic Relation
Boundary conditions	B.C.1 x=0 $T = T_o$ or $\theta = \theta_o = (T_o - T_\infty)$ B.C.2 x=L $\frac{dT}{dx} = -\frac{L}{k}(T_L - T_\infty)$ or $\frac{d\theta}{dx} = -\frac{h}{k}\theta_L$
T.D.E	$\frac{\theta}{\theta_o} = \frac{T - T_\infty}{T_o - T_\infty} = \frac{\cosh(m(L-x)) + \frac{h}{km}\sinh(m(L-x))}{\cosh(mL) + \frac{h}{km}\sinh(mL)}$
Heat transfer	$\dot{Q}_{fin} = \sqrt{hPkA}\theta_o \frac{\sinh(mL) + \frac{h}{km}\cosh(mL)}{\cosh(mL) + \frac{h}{km}\sinh(mL)}$ $\dot{Q}_{fin} = \sqrt{hPkA}(T_o - T_\infty)\frac{\sinh(mL) + \frac{h}{km}\cosh(mL)}{\cosh(mL) + \frac{h}{km}\sinh(mL)}$
Fin efficiency η_f	$\eta_{fin} = \frac{1}{mL}\frac{\sinh(mL) + \frac{h}{km}\cosh(mL)}{\cosh(mL) + \frac{h}{km}\sinh(mL)}$
Fin effectiveness ε_f	$\varepsilon_{fin} = \sqrt{\frac{Pk}{A_o h}}\frac{\sinh(mL) + \frac{h}{km}\cosh(mL)}{\cosh(mL) + \frac{h}{km}\sinh(mL)}$

- Example: A fin of material with thermal conductivity (100W/m.^{o}C). The length of the fin is 0.5m. The fin of square cross section of side length 0.1m. The base temperature of the fin is $100^{o}C$. The surrounding temperature and convection heat transfer coefficient are $20^{o}C$ and 20W/ $m^{2}.^{o}C$. Find the temperature Distribution equation and draw that with length of fin. Find the heat transfer and fin efficiency and effectiveness. By 1) assuming the fin of infinite length, 2) insulted end fin, 3) non-insulted fin.

- **Solution:** Fin of length L=0.5m, It is of square cross section of side length b=0.1m the P=4b=0.4m and A=$b^2 = 0.1 \times 0.1 = 0.01m^2$.
- The base temperature $T_o = 100^o C$.
- The surrounding temperature $T_\infty = 20^o C$
- The heat transfer coefficient $h = 10W/m^2.^o C$
- Thermal conductivity of fin material k=100W/m.$^o C$
- **Properties:** The properties are constant.
- **Assumption:** straight constant cross-sectional area fin with steady state heat conduction.
- **Analysis**: for the fin, we can determined firstly m.
- $$m = \sqrt{\frac{hP}{kA}} = \sqrt{\frac{10\times0.4}{100\times0.01}} = 2m^{-1}$$

- 1- Long Fin

The T.D.E $\frac{T-T_\infty}{T_o-T_\infty} = exp\left(-\sqrt{\frac{hP}{kA}}x\right) = e^{-mx} = e^{-2x}$

- $\frac{T-20}{100-20} = \frac{T-20}{80} = e^{-2x}$
- 2- insulated fin
- $\frac{T-T_\infty}{T_o-T_\infty} = \frac{cosh[m(L-x)]}{cosh(mL)}$
- $\frac{T-20}{100-20} = \frac{cosh[2(0.5-x)]}{cosh(2\times0.5)}$
- Convective end tip fin
- $\frac{T-T_\infty}{T_o-T_\infty} = \frac{\cosh(m(L-x))+\frac{h}{km}sinsh(r}{\cosh(mL)+\frac{h}{km}sinh(m}$
- $\frac{T-20}{100-20} = \frac{\cosh(2(0.5-x))+\frac{10}{200}\text{sinsh}(2(0.5-x))}{\cosh(0.5\times2)+\frac{10}{200}sinh(0.5\times2)}$

x	T(1)	T(2)	T(3)
0	100	100	100
0.05	92.39	94.30	93.29
0.1	85.50	89.34	88.23
0.15	79.26	85.07	83.85
0.2	73.62	81.46	80.10
0.25	68.52	78.46	76.96
0.3	63.90	76.05	74.39
0.35	59.72	74.20	72.36
0.4	55.9	72.88	70.46
0.45	52.52	72.10	69.87
0.5	49.43	71.84	69.37

- The heat transfer from the fin
- 1) long Fin $\dot{Q}_{fin} = \sqrt{hPkA}\theta_o = \sqrt{hPkA}(T_o - T_\infty)$
- $\dot{Q}_{fin} = \sqrt{10 \times 0.4 \times 100 \times 0.01}(100 - 20) = 160W$
- 2) insulated end tip fin
- $\dot{Q}_{fin} = \sqrt{hPkA}(T_o - T_\infty)\tanh(mL)$
- $\dot{Q}_{fin} = \sqrt{10 \times 0.4 \times 100 \times 0.01}(100 - 20)\tanh(2 \times 0.5)$
  $= 121.855\text{W}$
- 3) convective end fin
- $\dot{Q}_{fin} = \sqrt{hPkA}(T_o - T_\infty)\dfrac{sinh(mL)+\frac{h}{km}\cosh(mL)}{\cosh(mL)+\frac{h}{km}sinh(mL)}$
- $\dot{Q}_{fin} = 160\dfrac{sinh(1)+0.05\cosh(1)}{\cosh 1+0.05sinh(1)} = 125.1W$

- **Fin efficiency**
- 1) long fin $\eta_{fin} = \frac{1}{mL} = \sqrt{\frac{kA}{hP}\frac{1}{L}}$
- $\eta_{fin} = \frac{1}{mL} = \sqrt{\frac{100\times0.01}{10\times0.4}\frac{1}{0.5}} = 100\%$
- 2) insulated tip fin $\eta_{fin} = \frac{\tanh(mL)}{mL}$
- $\eta_{fin} = \frac{\tanh(2\times0.5)}{2\times0.5} = 76.16\%$
- 3) convective end fin $\eta_{fin} = \frac{1}{mL}\frac{sinh(mL)+\frac{h}{km}\cosh(mL)}{\cosh(mL)+\frac{h}{km}sinh(mL)}$
- $\eta_{fin} = \frac{1}{2\times0.5}\frac{sinh(1.0)+\frac{10}{2\times100}\cosh(1.0)}{\cosh(1.0)+\frac{10}{2\times1000}sinh(1.0)} = 78.18\%$

- **Fin effectiveness**
- 1) long Fin $\varepsilon_{fin} = \sqrt{\frac{Pk}{A_o h}} = \sqrt{\frac{0.4\times 100}{0.01\times 10}} = 20$
- 2) Insulated tip fin $\varepsilon_{fin} = \sqrt{\frac{0.4\times 100}{0.1\times 10}}\tanh(2\times 0.5)$
- ε_{fin} =15.23
- 3) convective end tip fin
- $\varepsilon_{fin} = \sqrt{\frac{Pk}{A_o h}}\frac{sinh(mL)+\frac{h}{km}\cosh(mL)}{\cosh(mL)+\frac{h}{km}sinh(mL)}$
- $\varepsilon_{fin} = 20\frac{sinh(1.0)+0.05\cosh(1.0)}{\cosh(1.0)+0.05sinh(1.0)} = 15.30$