Ordinary Differential Equation

Definition of a Differential Equation

A differential equation is an equation involving derivatives or differentials. The following are some examples of differential equations.

Ex. 1. $(y'')^2 + 3x = 2(y')^3$ where $y' = dy/dx$, $y'' = d^2y/dx^2$

Ex. 2. $\frac{dy}{dx} + \frac{y}{x} = y^2$

Ex. 3. $\frac{d^2Q}{dt^2} - 3\frac{dQ}{dt} + 2Q = 4\sin(2t)$

Ex. 4. $\frac{dy}{dx} = \frac{x+y}{x-y}$ or $(x + y)\,dx + (y - X)\,dy = 0$

Ex. 5. $\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0$

Equations such as those in examples (1 – 4) involving only one independent variable are called ***ordinary differential equations***. Equation such as that of example 5 or more independent variables are called ***partial differential equations***.

Order and degree of a differential equation

Order: (highest derivative), *Degree:* (Power of highest derivative)

Ex. 6.

1- $x^2\bar{\bar{y}} + \bar{y} + (x^2 - 4)y = 0$ — Ordinary, Order 2, Degree 1

2- $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 + \frac{y}{x^2+1} = e^x$ — Ordinary, Order 3, Degree 2

3- $\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$ — Partial, Order 4, Degree 1

4- $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ — Partial, Order 2, Degree 1

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3- Solution of first order ordinary differential equations (O.D.E.)

1- First order: variable separable

A first order differential equation can be solved by integration if it is possible to collect all y-terms with dy and all x-terms with dx, that is, if it is possible to write the equation in the form:

$$M(x)\, dx + N(y)\, dy = 0$$

than the general solution:

$$\int M(x)\, dx + \int N(y)\, dy = c$$

Where:

c is an arbitrary constant

Ex. 1. : Solve the differential equation:-

$$\frac{dy}{dx} = (1 + y^2)e^x$$

Solution

$$e^x dx - \frac{1}{1+y^2} dy = 0$$

$$\int e^x dx - \int \frac{dy}{1+y^2} = \int 0$$

$$e^x - \tan^{-1} y = c$$

$$\tan^{-1} y = e^x - c$$

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$$y = \tan(e^x - c)$$

Ex. 2. : Solve the differential equation:-

$$\boldsymbol{\frac{dy}{dx} = \frac{1}{x} y = 0}$$

Solution

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

$$\frac{dy}{y} + \frac{dx}{x} = 0$$

$$\int \frac{dy}{y} + \int \frac{dx}{x} = c$$

$$\ln y + \ln x = c$$

$$\ln xy = c$$

$$yx = e^c$$

$$y = \frac{e^c}{x}$$

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Ex. 3. : Solve the differential equation:-

$$\boldsymbol{\frac{dy}{dx} = e^{x-y}}$$

Solution

$$\frac{dy}{dx} = \frac{e^x}{e^y}$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + c$$

$$\ln e^y = \ln(e^x + c)$$

y= $\ln(e^x + c)$

***H.W.*: Solve the differential equation:-**

1) $\boldsymbol{x\frac{dy}{dx} + 3y = 2}$ **Given y = 0 , x = 0**

2) $x(2y - 3)\,dx + (x^2 + 1)\,dy = 0$

3) $x^2(y^2 + 1)\,dx + y\sqrt{x^3 + 1}\;dy = 0$

4) $x^2(y + 1)\,dx + y^2(x - 1)\,dy = 0$

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2 - First order: homogenous

An equation of the form:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \text{.... (1)}$$

Is called a homogenous equation. In this equation, the variable cannot be separated but can be transformed by a change of variable into an equation where the variables can be separated.

To transform a homogenous equation such equation (1) into an equation whose variable may separate,

We introduce the new independent variable:

$$v = \frac{y}{x}, \quad \frac{1}{v} = \frac{x}{y} \quad \text{..........(2)}$$

$$y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

substitute in (1) $$v + x\frac{dv}{dx} = F(v)$$

this equation is a variable separable $$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$

Then after solving this equation, the original eq. maybe obtained by substituting $\frac{y}{x}$ for v.

Ex. 1.: Show that the equation is homogenous and find solutions satisfied condition y(1) = 1:-

$$\frac{dy}{dx} = \frac{-(x^2 + y^2)}{2xy}$$

Solution

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$$\frac{dy}{dx} = -\left(\frac{\frac{x^2}{x^2} + \frac{y^2}{x^2}}{\frac{2xy}{x^2}}\right) = -\left(\frac{1 + \left(\frac{y}{x}\right)^2}{\frac{2y}{x}}\right)$$

$$\frac{dy}{dx} = -\left(\frac{1 + v^2}{2v}\right) = F(v)$$

$$\frac{dy}{x} + \frac{dv}{v - \left(-\left(\frac{1+v^2}{2v}\right)\right)} = 0$$

$$\frac{dy}{x} + \frac{dv}{v + \left(\frac{1 + v^2}{2v}\right)} = 0$$

$$\frac{dy}{x} + \frac{dv}{\frac{2v^2 + 1 + v^2}{2v}} = 0$$

$$\frac{dy}{x} + \frac{2vdv}{1 + 3v^2} = 0$$

$$\int x^{-1}dx + \frac{1}{3}\int (1 + 3v^2)^{-1} . 6vdv = \int 0\, dx$$

$$\ln x + \frac{1}{3}\ln(1 + 3v^2) = c$$

$$3\ln x + \ln(1 + 3v^2) = 3c$$

$$\ln x^3 \left(1 + 3v^2\right) = 3c$$

$$x^3 \left(1 + 3\frac{y^2}{x^2}\right) = e^{3c} = c' \qquad \text{Let} \quad c' = e^{3c}$$

$$x^3 + 3xy^2 = c'$$

$$(1)^3 + 3(1)(1)^2 = c'$$

$$c' = 1 + 3 = 4$$

$$x^3 + 3xy^2 = 4$$

Ex. 2.: Solve the differential equation:-

$$\boldsymbol{(x^2 + y^2)dx + xydy = 0}$$

Solution

$$xydy = -(x^2 + y^2)dx$$

$$\frac{dy}{dx} = \frac{-(x^2 + y^2)}{xy} = \frac{-\left(\frac{x^2}{x^2} + \frac{y^2}{x^2}\right)}{\frac{xy}{x^2}} = \frac{-\left(1 + \left(\frac{y}{x}\right)^2\right)}{\frac{y}{x}}$$

$$\frac{dy}{dx} = \frac{-(1 + v^2)}{v} = F(v)$$

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$

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$$\frac{dx}{x} + \frac{dv}{v - \left(\frac{-(1+v^2)}{v}\right)} = 0$$

$$\frac{dx}{x} + \frac{dv}{v + \frac{1+v^2}{v}} = 0$$

$$\frac{dx}{x} \frac{dv}{\frac{v^2+1+v^2}{v}} = 0$$

$$\frac{dx}{x} + \frac{vdv}{2v^2+1} = 0$$

$$\int x^{-1}dx + \frac{1}{2}\int (2v^2+1)^{-1}.2vdv = \int 0\,dx$$

$$\ln x + \frac{1}{2}\ln(2v^2+1) = c$$

$$2\ln x + \ln(2v^2+1) = 2c$$

$$\ln(x^2)(2v^2+1) = 2c$$

$$\ln(x^2)\left(\frac{2y^2}{x^2}+1\right) = 2c$$

$$\ln(2y^2+x^2) = 2c$$

$$2y^2 + x^2 = e^{2c}$$

$$2y^2 + x^2 = c'$$

H.W.: Solve the differential equation:-

1) $(x^2 + y^2)\, dx + xy\, dy = 0$

2) $x^2\, dy + (y^2 - xy)\, dx = 0$

3) $x\, e^{y/x}\, dx + y dx - x dy = 0$

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