Linear second order homogeneous equations with constant coefficients:

An equation of the form:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + a_n \frac{dy}{dx} + a_n y = F(x)$$

Which is linear in y and its derivatives, is called a **linear** equation of order n. if $F(x)$ is identically zero, the equation is said to be **homogeneous**, otherwise it is called **nonhomogeneous**. The equation is called linear even when the coefficients $a_1, a_2, \cdots, a_n$ are functions of x . however, we shall consider only the case where these coefficient are **constants.**

Linear differential operators:

At this point, it is convenient to introduce the symbol D to represent the operation of differentiation with respect to x. that is, we write $Df(x)$ to mean $(d/dx)f(x)$. Furthermore, we define powers of D to mean taking successive derivatives:

$$D^2 f(x) = D\{Df(x)\} = \frac{d^2 f(x)}{dx^2}$$

$$D^3 f(x) = D\{D^2 f(x)\} = \frac{d^3 f(x)}{dx^3}$$

The characteristic equation:

In the remainder of this article, we consider only linear second order equations with constant coefficients. Because the solutions of nonhomogeneous equations depend on the solutions of the corresponding homogeneous equations, we focus on equations of the form

$$(D^2 + 2aD + b) = 0 \qquad \ldots\ldots\ldots\ldots (1)$$

The usual method of solving equation (1) is to begin by factoring the operator, thus

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$$(D^2 + 2aD + b) = (D - r_1)(D - r_2)$$

We do this by finding the two roots r_1 and r_2 of the equation:

$$r^2 + 2ar + b = 0 \qquad \ldots\ldots\ldots\ldots (1')$$

The equation $(1')$ is called the **characteristic equation** of the differential equation.

***Case*1:**

$$\boxed{y = C_1 e^{r_1 x} + C_2 e^{r_2 x}} \qquad \text{If } r_1 \neq r_2$$

***Case*2:**

$$\boxed{y = (C_1 x + C_2) e^{r_2 x}} \qquad \text{If } r_1 = r_2$$

***Case*3:**

$$\boxed{y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)} \qquad \text{if} \qquad \begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

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Example: Solve the equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$$

Solution:

$r^2+r-2=0$

$(r-1)(r+2)=0$

$r_1=1$

$r_2=-2$

$r_1\neq r_2$

$\therefore\ y=C_1e^{r_1x}+C_2e^{r_2x}$

$y=C_1e^{x}+C_2e^{-2x}$

Example: Solve the equation

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$$

Solution:

$r^2+4r+4=0$

$(r+2)^2=0$

$r_1=r_2=-2$

$r_1=r_2$

$\therefore\ y=(C_1x+C_2)e^{r_2x}$

$y=(C_1x+C_2)e^{-2x}$

Example: Solve the equation

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0$$

Solution:

$r^2+2r+2=0$

$r_1=-1+i$

$r_2=-1-i$

$\therefore\ \alpha=-1 \qquad , \quad \beta=1$

$y=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$

$y=e^{-x}(C_1\cos x+C_2\sin x)$

H.W: Solve the following equations:

1. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$

2. $\frac{d^2y}{dx^2}-2\frac{dy}{dx}-3y=0$

3. $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=0$

4. $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+5y=0$

Linear second order nonhomogeneous equations with constant coefficients:

We learned how to solve the homogeneous equation

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=0 \qquad \text{.......... (1)}$$

We can now describe a method for solving the nonhomogeneous equation

$$\frac{d^2y}{dx^2}+2a\frac{dy}{dx}+by=F(x) \qquad \text{.......... (2)}$$

To solve equation (2), we first determine the general solution of the related homogeneous equation (1), obtained by replacing $F(x)$ by zero. Let us denote this solution by

$$y_h = C_1u_1(x)+C_2u_2(x)$$

Where C_1 and C_2 are arbitrary constants and $u_1(x)$, $u_2(x)$ are functions of one or more of the following forms:

$$e^{\alpha x}\sin\beta x \;\; e^{rx} \;,\; xe^{rx}, e^{\alpha x}\cos\beta x \;,$$

These ways involve finding *a particular solution* of equation (2) and adding it to the general solution of equation (1). We shall discuss three

These ways involve finding *a particular solution* of equation (2) and adding it to the general solution of equation (1). We shall discuss three
ways of finding a particular solution:

1. *inspired guessing.*
2. *the method of variation of parameters.*
3. *the method of undetermined coefficients.*

But first, let us verify that all solutions of the nonhomogeneous equation are of the form:

$$\boxed{y = y_h(x)+y_p(x)}$$

Where:

$y_h(x)$: is the general solution of the homogeneous equation (1).

$y_p(x)$: is any particular solution of the nonhomogeneous equation (2).

Inspired guessing:
In the following example , it does not require a genius to guess that three may be a particular solution of the form $y_p(x) =$ constant. If there is such a solution, the particular constant that works can easily be determined from the equation.

Example: Solve the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6$$

Solution: the solution y_h satisfies the homogeneous equation

$$\frac{d^2y_h}{dx^2} + 2\frac{dy_h}{dx} - 3y_h = 0$$

It characteristic equation is:

$$r^2 + 2r - 3 = 0$$
$$(r+3)(r-1) = 0$$

$$r_1 = -3$$

$$r_2 = 1$$

$$r_1 \neq r_2$$

$\therefore$

$$y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$y_h = C_1 e^{-3x} + C_2 e^{x}$$

Now, to find a particular solution of the original equation, observe that $y =$ constant would do, provided $-3y = 6$. hence

$$y_p = -2$$

Is one particular solution. The complete solution is:

$$y = y_h + y_p$$

$$y = C_1 e^{-3x} + C_2 e^{x} - 2 \therefore$$

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Nonlinear equation (Bernoulli Equation)

A first-order differential equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \qquad n \in \mathbb{R}$$

is called a Bernoulli differential equation.

To solve this equation, must nonlinear equation **convert** to linear equation.

The method:

1- Divide two side of equation by $\mathbf{y^n}$

$$\frac{dy}{dx} + P(x)\,y = Q\,(x)\,y^n \qquad \div\, y^n$$

2- Submit

$$Z = y^{1-n} \qquad \frac{dz}{dx} = (1-n)\,y^{-n}$$

To convert to linear equation.

***Note**: It can find the equation in the* $\frac{dx}{dy} + P(y)\,x = Q\,(y)\,x^n$

Ex. 13. Solve

1- $2\frac{dy}{dx} - \frac{y}{x} = -y^3 \cos x$ (nonlinear) $\div\, y^3$

$$2y^{-3}\frac{dy}{dx} - \frac{1}{x}\,y^{-2} = -\cos x \qquad \text{let } z = y^{1-n} = y^{1-3} = y^{-2}$$

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \qquad -\frac{dz}{dx} - \frac{1}{x}\,z = -\cos x \qquad * -1$$

$$\frac{dz}{dx} + \frac{1}{x}\,z = \cos x \qquad \text{linear equation}$$

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$P = \frac{1}{x}$, $Q = \cos x$ $\rho = e^{\int P(x)dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$\rho.z = \int \rho\, Q(x)\, dx$ $x.z = \int x . \cos x \, dx$

Let $u = x$ $du = dx$

$dv = \cos x \, dx$ $v = \sin x$

$\int u dv = uv - \int v du$

$x.z = x. \sin x - \int \sin x \, dx$

$x.z = x. \sin x + \cos x + c$

$x. y^{-2} = x \sin x + \cos x + c$

H.W. Solve the following equations

1) 3xy′ + y + x2y4 = 0

2) y2y′ + x2y3 = x2

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2- $3xy' + y + x^2y^4 = 0$ $\qquad \div 3x$

$$y' + \frac{1}{3x}y + \frac{x}{3}y^4 = 0$$

$$y' + \frac{1}{3x}y = -\frac{x}{3}y^4 \qquad \div y^4$$

$$y^{-4}y' + \frac{1}{3x}y^{-3} = -\frac{x}{3}$$

let $z = y^{-3}$ $\qquad \frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{y^4}{3}\frac{dz}{dx}$$

$$y^{-4}\left(-\frac{y^4}{3}\right)\frac{dz}{dx} + \frac{1}{3x}z = -\frac{x}{3} \qquad -\frac{1}{3}\frac{dz}{dx} + \frac{1}{3x}z = -\frac{x}{3} \qquad * -3$$

$$\frac{dz}{dx} - \frac{1}{x}z = x \qquad \text{linear}$$

$P = \frac{1}{x}$, $Q = x$ $\qquad \phi = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$$\phi.z = \int \phi\, Q\, dx \qquad \frac{1}{x}\,.\,y^{-3} = \int \frac{1}{x}\,.\,x\, dx$$

$$\frac{1}{x}\,y^{-3} + x + c$$

3- $y^2y' + x^2y^3 = x^2$

Let $z = y^3$

$\frac{dz}{dx} = 3y^2\frac{dy}{dx}$ $\qquad\qquad$ $\frac{dy}{dx}y^2 = \frac{1}{3}\frac{dz}{dx}$

$\frac{1}{3}\frac{dz}{dx} + x^2 z = x^2$ $\qquad\qquad$ $* 3$

$\frac{dz}{dx} + 3x^2 z = 3x^2$ $\qquad\qquad$ $P = 3x^2$ $\qquad$ $Q = 3x^2$

$\phi = e^{\int P(x)dx} = e^{\int 3x^2 dx} = e^{x^3}$

$\phi . z = \int \phi\, Q\, dx$ $\qquad\qquad$ $e^{x^3} . z = \int e^{x^3} . 3x^2\, dx$

$e^{x^3} . z = e^{x^3} + c$ $\qquad\qquad$ $e^{x^3} . y^3 = e^{x^3} + c$

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