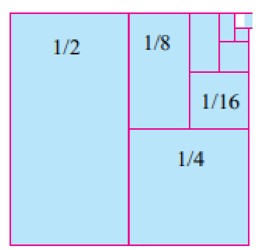
INFINITE SEQUENCES AND SERIES

OVERVIEW While everyone knows how to add together two numbers, or even several, how to add together infinitely many numbers is not so clear. In this chapter we study such questions, the subject of the theory of infinite series. Infinite series sometimes have a finite sum, as in

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots \ldots \ldots \ldots \ldots = 1$$

This sum is represented geometrically by the areas of the repeatedly halved unit square shown here. The areas of the small rectangles add together to give the area of the unit square, which they fill. Adding together more and more terms gets us closer and closer to the total.



Other infinite series do not have a finite sum, as with

$$1 + 2 + 3 + 4 + 5 + \ldots \ldots .$$

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11.1, SEQUENCES

A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots\ldots$$

in a given order. Each of a_1, a_2, a_3 and so on represents a number. These are the **terms** of the sequence. For example the sequence

$$2, 4, 6, 8, 10, 12\ldots 2n\ldots$$

has first term $a = 2$, second term $a_2 = 4$ and nth term $a_n = 2n$. The integer n is called the **index** of a_n, and indicates where a_n occurs in the list. We can think of the sequence

$$a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$$

As a function that sends 1 to a_1, 2 to a_2, 3 to a_3, and in general sends the positive integer n
To the nth term a_n. This leads to the formal definition of a sequence.

in which a and r are fixed real numbers and $a \neq 0$. The series can also be written as $\sum_{n=0}^{\infty} ar^n$. The **ratio** r can be positive, as in

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots,$$

or negative, as in

$$1 - \frac{1}{3} + \frac{1}{9} - \cdots + \left(-\frac{1}{3}\right)^{n-1} + \cdots.$$

If $r = 1$, the nth partial sum of the geometric series is

$$s_n = a + a(1) + a(1)^2 + \cdots + a(1)^{n-1} = na,$$

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DEFINITION Infinite Sequence
An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

The function associated to the sequence

$$2, 4, 6, 8, 10, 12, \ldots, 2n, \ldots\ldots$$

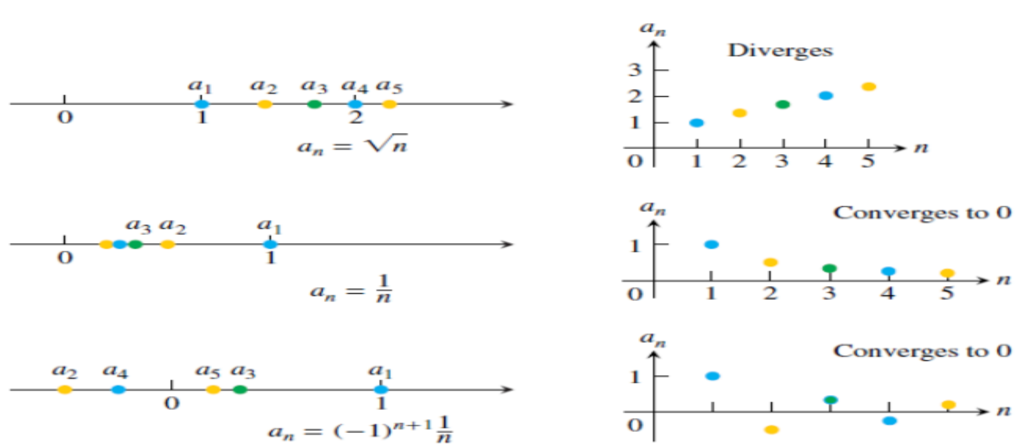
Sends 1 to $a_1 = 2$, 2 to $a_2 = 4$, and so on. The general behavior of this sequence is de- scribed by the formula

$$a_n = 2n.$$

Convergence and Divergence

Figure 11.1 shows two ways to represent sequences graphically. The first marks the first few points from $a_1, a_2, a_3, \ldots, a_n, \ldots\ldots$

on the real axis. The second method shows the graph of the function defining the sequence. The function is defined only on integer inputs, and the graph consists of some points in the xy-plane, located at $(2, a_2), \ldots, (n, a_n), \ldots$



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TEST FOR CONVERGENCE AND DIVERGENCE SERIES

1- The nth Term test (for Divergence)

If $\lim_{n\to\infty} an \neq 0$ ***or*** $\lim_{n\to\infty} an \rightarrow not\ exist \ \ggg \sum_{n=1}^{\infty} an$ ***diverges***

Note: ***If*** $\lim_{n\to\infty} an = 0 \ggg the\ test\ is\ fail$.

Examples :

Test the divergence for the following:

1- $\sum_{n=1}^{\infty} \frac{n+1}{n}$, $a_n = \frac{n+1}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \frac{1}{n} = 1 \quad Div.$$

2- $\sum_{n=1}^{\infty} (-1)^{n+1}$, $a_n = (-1)^{n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^{n+1} = +1, -1,$$
$$\}not\ exist \gg Div.$$

3- $\sum_{n=1}^{\infty} 2(\cos\frac{\pi}{3})^n$, $a_n = 2(\cos\frac{\pi}{3})^n = 2(\frac{1}{2})^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2(\frac{1}{2})^n = 2(0) = 0 \quad (May\ be\ converge,$$
$$therfore\ we\ need\ to\ other\ test.$$

Note: if $\lim_{n\to\infty} a_n = 0 \rightarrow \sum_{n=1}^{\infty} a_n \ \ may\ converge.$

Geometric Series:

A series of form:

$\sum_{n=1}^{\infty} a_n = a + ar + ar^2 + ar^3 + \cdots + a(r)^{n-1}$ is a geometric series

Where, r is positive or negative.

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^n$$

$$a = 1 \qquad r = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{3^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{-1}{3}\right)^{n-1} = 1 - \frac{1}{3} + \frac{1}{9}$$

$$a = 1 \quad r = -\frac{1}{3}$$

Convergence of geometric series:

1- If $|r| < 1$ *the geometric series converges to* $\frac{a}{(r-1)}$

2- If $|r| \geq 1$ *the geomtric series diverges unless* $a = 0$

Examples:

1- $\sum_{n=1}^{\infty} \frac{4}{2^{n-1}} = \sum_{n=1}^{\infty} 4(\frac{1}{2})^{n-1} = 4\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}$

$$a = 1 \qquad , \; r = \frac{1}{2} < 1$$

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$$\gg \sum_{n=1}^{\infty} \frac{4}{2^{n-1}} \quad is\ Conv. \qquad \gg \sum_{n=1}^{\infty} \frac{4}{2^{n-1}} = 4\frac{1}{1-\frac{1}{2}} = 8$$

2- $\sum_{n=0}^{\infty} \frac{3^n-2^n}{6^n} = \sum_{n=1}^{\infty} \frac{3^{n-1}-2^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1}$

$$\gg \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \quad \gg \ r = \frac{1}{2} < 1 \quad conv.$$

$$also \ \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} \quad \gg \ r = \frac{1}{3} < 1 \quad conv.$$

$$\gg \ \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{n-1} \quad \text{is conv.}$$

$$\frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = \frac{1}{2}$$

3- $\sum_{n=1}^{\infty} \frac{n}{n+1}$, $\qquad a_n = \frac{n}{n+1} \gg \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1 \qquad div.$

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$$\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$$

$$\frac{4}{(4n-3)(4n+1)} = \frac{A}{(4n-3)} + \frac{B}{(4n+1)} = \frac{A(4n+1)+B(4n-3)}{(4n-3)(4n+1)}$$

$$\frac{4}{(4n-3)(4n+1)} = \frac{4nA + A + 4nB - 3B}{(4n-3)(4n+1)}$$

$\gg (4A+4B)n = 0 \quad \ldots\ldots\ldots\ldots..(1)$

$\gg A - 3B = 4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

$$\ggg \sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \sum_{n=1}^{\infty} \frac{1}{(4n-3)} - \sum_{n=1}^{\infty} \frac{1}{(4n+1)}$$

$$= \left(1-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{9}\right) + \left(\frac{1}{9}-\frac{1}{13}\right) \ldots\ldots\ldots \left(\frac{1}{(4n-3)}\right) - \left(\frac{1}{(4n+1)}\right)$$

$$= 1 - \frac{1}{(4n+1)} = \frac{4n+1-1}{4n+1} = \frac{4n}{4n+1}$$

$$\lim_{n\to\infty} \frac{4n}{4n+1} = \lim_{n\to\infty} \frac{4}{4+\frac{1}{n}} = 1 \qquad conv.\ \ to\ \ 1$$

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