P- Series:

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots\dots\dots\dots\frac{1}{k^p}$$

$Convergence\ if\ \ p>1$

$Divergence\ \ if\ \ 0<p\leq 1$

Examples:

1- $\sum_{n=1}^{\infty}\ \frac{1}{n}$ $\qquad P-Series\ ,\ P=1\quad Div.$

2- $\sum_{n=1}^{\infty}\ \frac{1}{n^2}$ $\qquad P-Series\ ,\ P>1\quad Conv.$

3- $\sum_{n=1}^{\infty}\ \frac{1}{\sqrt{n}}$ $\qquad P-Series\ \ ,P<1\quad Div.$

2- Integral Series:

Examples:

1- $\sum_{n=1}^{\infty}\frac{1}{n^2}$ $\qquad P-Series\quad Conv.$

Can test it by Integral test:

$$\gg\int_1^{\infty}\frac{1}{x^2}\,dx\ =\ -\frac{1}{x}\ {}_1^{\infty}]=\lim_{L\to\infty}-\frac{1}{x}\ {}_1^{L}]=\frac{1}{1}-\frac{1}{L}=1$$

$The\ integral\ is\ conv.\gg the\ series\ is\ conv.$

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2- $\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad P - Series \quad Div.$

Can test it by Integral test:

$$\gg \int_1^{\infty} \frac{1}{x}\, dx = \ln x \,\Big]_1^{\infty} = \lim_{L\to\infty}(\ln x)\,\Big]_1^{L} = \lim_{L\to\infty}(\ln L) = \infty$$

$The\ integral\ is\ div. \gg the\ series\ is\ div.$

3-Ratio test :

(Positive term series)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

a) If $\rho < 1$ the series converges

b) If $\rho > 1$ the series diverges

c) If $\rho = 1$ the series may be conv. or div. another test must be tried.

ملاحظة: يستخدم هذا الاختبار غالبا عند احتواء الحدود على $n!$ او مقادير مرفوعة الى n

Ex.

1- $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\rho = \frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{\frac{1}{(n+1)n!}}{\frac{1}{n!}} = \frac{1}{n+1} = 0 \quad < 1 \ conv.$$

2- $\sum_{n=1}^{\infty} \frac{n}{2^n}$ $\quad by\ ratio\ test$

$$\rho = \lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2^n}{2.\,2^n}$$

$$\rho = \frac{1}{2} \lim_{n\to\infty} \frac{n+1}{n} = \frac{1}{2}(1) = \frac{1}{2} \quad < 1 \quad conv.$$

3- $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ *by ratio test*

$$\rho = \lim_{n\to\infty} \frac{\frac{1}{2(n+1)-1}}{\frac{1}{2n-1}} = \lim_{n\to\infty} \frac{1}{2(n+1)-1} . \frac{2n-1}{1}$$

$$= \lim_{n\to\infty} \frac{2n-1}{2n+1} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{2+\frac{1}{n}} = 1 \quad test\ is\ fail$$

$$Other\ test : by\ integral\ test$$

$$\int_1^{\infty} \frac{dx}{2x-1} = \lim_{L\to\infty} \left(\frac{1}{2} ln(2x-1)_1^L\right) = \frac{1}{2} \lim_{n\to\infty} \quad ln(2L-1) = \infty$$

series is div.

4- Root test

(Positive term series)

$$\rho = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} (a_n)^{\frac{1}{n}}$$

a) $If \quad \rho \ < 1 \quad the\ series\ converges$

b) $If \quad \rho \ > 1 \ or = \infty \quad the\ series\ diverges$

c) $If \quad \rho \ = 1 \quad the\ series\ may\ be\ conv.\ or\ div.$

Examples:

1- $\sum_{n=1}^{\infty}\left(\frac{4n-5}{2n+1}\right)^{n}$ *by root test*

$$\rho=\lim_{n\to\infty}\left[\left(\frac{4n-5}{2n+1}\right)^{n}\right]^{\frac{1}{n}}=\lim_{n\to\infty}\left[\left(\frac{4n-5}{2n+1}\right)\right]=\lim_{n\to\infty}\left[\frac{4-\frac{5}{n}}{2+\frac{1}{n}}\right]=2\ >1\ \ div.$$

2- $\sum_{n=1}^{\infty}\frac{1}{(\ln(n+1))^{n}}$ *by root test*

$$\rho=\lim_{n\to\infty}\left[\left(\frac{1}{\ln(n+1)}\right)^{n}\right]^{\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{\ln(n+1)}=0\ <1\quad conv.$$

5- Comparison test :

$$\textit{If } b_n > a_n \textit{ and } \sum b_n \textit{ was converges}$$

$$\gg \sum a_n \quad \textit{is converges}$$

$$\textit{If } b_n < a_n \textit{ and } \sum b_n \textit{ was diverges} \gg \sum a_n \quad \textit{is diverges}$$

$sin^2 \le 1$
$cos^2 \le 1$
$|sin| \le 1$
$|cos| \le 1$
$\ln n > 1$
$\ln n < n$
$n^{\frac{1}{2}} < n$
$n^{\frac{2}{5}} < n^{\frac{3}{2}}$

Examples:

1- $\sum_{n=1}^{\infty} \frac{\sin^2}{n^4}$ *Comparison test*

$\sin^2 < 1 \quad] \div n^4$

$\frac{\sin^2}{n^4} < \frac{1}{n^4}$ $\quad b_n = \frac{1}{n^4} \;\dots.\; P - series$

$P = 4 \quad > 1 \quad conv.$

$\therefore \frac{\sin^2}{n^4} \quad conv.$

2- $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^{\frac{3}{2}}}$

$|\cos n| \leq 1 \quad] \div n^{\frac{3}{2}}$

$\frac{|\cos n|}{n^{\frac{3}{2}}} \leq \frac{1}{n^{\frac{3}{2}}}$ $\quad b_n = \frac{1}{n^{\frac{3}{2}}} \;\dots.\; P - series$

$P = \frac{3}{2} \quad < 1 \quad div \quad .$

6- Limit comparison test :

Let $\sum a_n$ *and* $\sum b_n$ *be series with positive term and*

suppose $\quad \rho = \lim_{n\to\infty} \frac{a_n}{b_n}$

If ρ *is finit and* $\rho \neq 0$ *then the sereis both conv. or div.*

Examples:

1- $\sum_{n=1}^{\infty} \frac{1}{2n^2-n}$ *Limit comparison*

$let\ b_n = \frac{1}{n^2}$ $\quad P - series \rightarrow P > 1\ conv.$

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$$\rho = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2n^2-n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{2n^2-n} \qquad] \div n^2$$

$$= \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2} \qquad \to \rho \neq 0 \quad \therefore \text{the series is conv.}$$

2- $\sum_{n=1}^{\infty} \dfrac{1}{n-\frac{1}{4}}$ $\quad$ *Limit comparison*

$$Let\ b_n = \sum_{n=1}^{\infty} \frac{1}{n} \qquad P-series \ \to P = 1 \quad div.$$

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