Inverse Laplace Transform

If the Laplace transform of a function F(t) is f(s), i.e. if {F(t)} = f(s), then F(t) is called an inverse Laplace transform f(s) and it is symbolically expressed as follows:

$$F(t) = \mathcal{L}^{-1}\{f(s)\}$$

where

$\mathcal{L}^{-1}$ is called the inverse Laplace transformation operator.

Table of Laplace Transforms

f(t)	$F(s) = \mathscr{L}\{f\}(s)$
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > 0$
t^n, $n = 1,2,\ldots$	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}$, $s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}$, $s > 0$
$e^{at}t^n$, $n = 1,2,\ldots$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$

(Inverse Laplace Transform)

Example

Find $f(t)$ if

(a) $F(s)=\dfrac{5}{s+3}$, (b) $F(s)=\dfrac{s+1}{s^2+1}$, (c) $F(s)=\dfrac{1}{(s+25)^2}$,

(d) $F(s)=\dfrac{s+2}{(s+2)^2+1}$, (e) $F(s)=\dfrac{s}{(s-1)^2-4}$, (f) $F(s)=\dfrac{1}{s^2(s^2+1)}$,

(g) $F(s)=\dfrac{4}{s^2+2s+10}$

Solution

(a) $f(t)=5e^{-3t}$

(b) $F(s)=\dfrac{s+1}{s^2+1}=\dfrac{s}{s^2+1}+\dfrac{1}{s^2+1} \quad\Rightarrow\quad f(t)=\cos(t)+\sin(t)$

(c) Using the shifting property $\mathcal{L}\{e^{at}f(t)\}=F(s-a)$ then $\mathcal{L}^{-1}\{F(s-a)\}=e^{at}f(t)$.

Here, we have $F(s)=\dfrac{1}{s^2}$ with $a=-25$. Since, $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\}=t$ then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+25)^2}\right\}=t\cdot e^{-25t}$$

(d) Here, we have a shifting of -2 with $F_1(s)=\dfrac{s}{s^2+1}$. So, $f_1(t)=\cos(t)$ and

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\}=f(t)=\cos(t)\cdot e^{-2t}$$

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(Inverse Laplace Transform)

(e) $F(s)=\dfrac{s}{(s-1)^2-4}=\dfrac{s-1+1}{(s-1)^2-4}=\dfrac{s-1}{(s-1)^2-4}+\dfrac{1}{(s-1)^2-4}$

So, $f(t)=e^{t}\cosh(2t)+\dfrac{1}{2}e^{t}\sinh(2t)$

(f) We know that $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\}=\sin(t)$ and using the property of division by s which means an integration in time domain, we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+1}\right\}=\int_0^t \sin(u)du=1-\cos(t)$$

Again using the same property we get

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+1}\right\}=\int_0^t (1-\cos(u))du=t-\sin(t)$$

(g) $F(s)=\dfrac{4}{s^2+2s+10}=\dfrac{4}{s^2+2s+1-1+10}=\dfrac{4}{(s+1)^2+9}$

This is $F_1(s)=\dfrac{4}{s^2+9}$ with a shifting of $a=-1$. So, $f_1(t)=\dfrac{4}{3}\sin(3t)$ and

$f(t)=\dfrac{4}{3}e^{-t}\sin(3t)$

Solution

$$\frac{3s+7}{(s+1)(s-3)}=\frac{A}{s+1}+\frac{B}{s-3}$$

$$\Rightarrow \quad 3s+7=A(s-3)+B(s+1)$$

First Method

$$3s+7=As-3A+Bs+B \quad \Rightarrow A+B=3$$

$$\Rightarrow -3A+B=7$$

Solving these two equations we get $A=-1$ and $B=4$

Second Method

$$3s+7=A(s-3)+B(s+1)$$

At $s=-1$ we get $-3+7=-4A \Rightarrow A=-1$

At $s=3$ we get $9+7=4B \Rightarrow B=4$

Third Method

$$A=\left.\frac{3s+7}{(s-3)}\right|_{s=-1}=\frac{3(-1)+7}{-1-3}=\frac{4}{-4}=-1$$

$$B=\left.\frac{3s+7}{(s+1)}\right|_{s=3}=\frac{3(3)+7}{3+1}=\frac{16}{4}=4$$

$$F(s)=\frac{3s+7}{(s+1)(s-3)}=\frac{-1}{s+1}+\frac{4}{s-3}$$

$$f(t)=-e^{-t}+4e^{3t}$$

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Example

Find $f(t)$ if $F(s)=\dfrac{s-1}{(s+1)^3}$

Solution

$$\frac{s-1}{(s+1)^3}=\frac{A}{(s+1)}+\frac{B}{(s+1)^2}+\frac{C}{(s+1)^3}$$

$$C=s-1\Big|_{s=-1}=-2$$

$$B=\frac{1}{1!}\frac{d}{ds}(s-1)\Big|_{s=-1}=1$$

$$A=\frac{1}{2!}\frac{d^2}{ds^2}(s-1)\Big|_{s=-1}=\frac{1}{2}\frac{d}{ds}(1)\Big|_{s=-1}=0$$

$$\text{So } F(s)=\frac{1}{(s+1)^2}-\frac{2}{(s+1)^3}$$

$$\Rightarrow f(t)=e^{-t}\left(t-t^2\right)$$