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Class 3rd
(Engineering Analysis)
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Laplace Transform Theorem (Properties)

Theorem 1: Linearity

The Laplace transform is a linear transform, by which is meant that:

(1) *The transform of a sum (or difference) of expressions is the sum (or difference) of the individual transforms. That is*

$$L\{f(t)\pm g(t)\}=L\{f(t)\}\pm L\{g(t)\}$$

(2) *The transform of an expression that is multiplied by a constant is the constant multiplied by the transform of the expression. That is*

$$L\{kf(t)\}=kL\{f(t)\}$$

Note: Two transforms must **not** be multiplied together to form the transform of a product of expressions – we shall see later that the product of two transforms is the transform of the *convolution* of two expressions.

Example 6

(a) $L\{2e^{-t}+t\}=L\{2e^{-t}\}+L\{t\}=2L\{e^{-t}\}+L\{t\}=\dfrac{2}{s+1}+\dfrac{1}{s^2}=\dfrac{2s^2+s+1}{s^2(s+1)}$

(b) $L\{2\sin 3t+\cos 3t\}=2L\{\sin 3t\}+L\{\cos 3t\}=2.\dfrac{3}{s^2+9}+\dfrac{s}{s^2+9}=\dfrac{s+6}{s^2+9}$

(c) $L\{4e^{2t}+3\cosh 4t\}=4L\{e^{2t}\}+3L\{\cosh 4t\}$

$$=4.\frac{1}{s-2}+3.\frac{s}{s^2-16}=\frac{4}{s-2}+\frac{3s}{s^2-16}$$

$$=\frac{7s^2-6s-64}{(s-2)(s^2-16)}$$

The working is straightforward. **for**

(d) $L\{2\sin 3t+4\sinh 3t\}=2.\dfrac{3}{s^2+9}+4.\dfrac{3}{s^2-9}$

$$=\frac{6}{s^2+9}+\frac{12}{s^2-9}=\frac{18(s^2+3)}{s^4-81}$$

(e) $L\{5e^{4t}+\cosh 2t\}=\dfrac{5}{s-4}+\dfrac{s}{s^2-4}=\dfrac{6s^2-4s-20}{(s-4)(s^2-4)}$

(f) $L\{t^3+2t^2-4t+1\}=\dfrac{3!}{s^4}+2.\dfrac{2!}{s^3}-4.\dfrac{1!}{s^2}+\dfrac{1}{s}$

$$=\frac{1}{s^4}\{s^3-4s^2+4s+6\}$$

Theorem 2 The first shift theorem

The first shift theorem states that if $L\{f(t)\} = F(s)$ then

$L\{e^{-at}f(t)\} = F(s+a)$

Because $L\{e^{-at}f(t)\} = \int_{t=0}^{\infty} e^{-at}f(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{\infty} f(t)e^{-(s+a)t}\,\mathrm{d}t = F(s+a)$

That is $\quad L\{e^{-at}f(t)\} = F(s+a)$

The transform $L\{e^{-at}f(t)\}$ is thus the same as $L\{f(t)\}$ with s everywhere in the result replaced by $(s+a)$.

For example $L\{\sin 2t\} = \dfrac{2}{s^2+4}$

then $\quad L\{e^{-3t}\sin 2t\} = \dfrac{2}{(s+3)^2+4} = \dfrac{2}{s^2+6s+13}$

Similarly, $L\{t^2\} = \dfrac{2}{s^3}$. $\therefore L\{t^2e^{4t}\}$ is the same with s replaced by $(s-4)$.

$\therefore \; L\{t^2e^{4t}\} = \dfrac{2}{(s-4)^3}$

Exercise

Determine the following.

1. $L\{e^{-2t}\cosh 3t\}$
2. $L\{2e^{3t}\sin 3t\}$
3. $L\{4te^{-t}\}$
4. $L\{e^{2t}\cos t\}$
5. $L\{e^{3t}\sinh 2t\}$
6. $L\{t^3e^{-4t}\}$

6. $L\{t^3\} = \dfrac{3!}{s^4}$ $\qquad \therefore\ L\{t^3 e^{-4t}\} = \dfrac{6}{(s+4)^4}$

Theorem 3 Multiplying by t and t^n

If $L\{f(t)\} = F(s)$ then $\quad L\{tf(t)\} = -F'(s)$

Because $L\{tf(t)\} = \int_{t=0}^{\infty} tf(t)e^{-st}\,dt = \int_{t=0}^{\infty} f(t)\left(-\dfrac{de^{-st}}{ds}\right) dt$

$$= -\frac{d}{ds}\int_{t=0}^{\infty} f(t)e^{-st}\,dt = -F'(s)$$

For example, $\quad L\{\sin 2t\} = \dfrac{2}{s^2+4}$

$$\therefore\ L\{t\sin 2t\} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \frac{4s}{(s^2+4)^2}$$

and similarly, $\quad L\{t\cosh 3t\} = -\dfrac{d}{ds}\left(\dfrac{s}{s^2-9}\right) = -\dfrac{(s^2-9)-s(2s)}{(s^2-9)^2}$

$$= \frac{s^2+9}{(s^2-9)^2}$$

Theorem 4 Division by t

If $F(s)=\mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\}=\int_s^\infty F(u)du$$

Example

$$\mathcal{L}\left\{\frac{\sin(t)}{t}\right\}$$

Here, $f(t)=\sin(t)$ then $F(s)=\frac{1}{s^2+1}$ and

$$\mathcal{L}\left\{\frac{\sin(t)}{t}\right\}=\int_s^\infty \frac{1}{u^2+1}du=\tan^{-1}(u)\Big|_s^\infty=\frac{\pi}{2}-\tan^{-1}(s)=\tan^{-1}\left(\frac{1}{s}\right)$$

$$L\{t^n f(t)\}=(-1)^n\frac{d^n}{ds^n}\{F(s)\}$$

Example 2

Determine $L\left\{\frac{1-\cos 2t}{t}\right\}$

$$L\{1-\cos 2t\}=\frac{1}{s}-\frac{s}{s^2+4}$$

Then, by Theorem 4

$$L\left\{\frac{1-\cos 2t}{t}\right\}=\int_{\sigma=s}^{\infty}\left\{\frac{1}{\sigma}-\frac{\sigma}{\sigma^2+4}\right\}d\sigma$$

$$=\left[\ln\sigma-\frac{1}{2}\ln(\sigma^2+4)\right]_{\sigma=s}^{\infty}=\frac{1}{2}\left[\ln\left(\frac{\sigma^2}{\sigma^2+4}\right)\right]_{\sigma=s}^{\infty}$$