Partial derivative

A partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant. The symbol used to denote partial derivatives is ∂ .

The partial derivative of a function $f(x,y,z)$ denoted with respect to the variable by: $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$

Ex: Find the partial derivative $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ of a function $f(x,y,z) = 3\,x^2\,y^4\,z^6$:

$\frac{\partial f}{\partial x} = (2x)\;3y^4\,z^6$ $\qquad$ $\frac{\partial f}{\partial y} = (4\,y^3)\;\;3\,x^2\,z^6$ $\qquad$ $\frac{\partial f}{\partial z} = (6z^5)3\,x^2\,y^4$

Ex: Find the partial derivative $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ of a function $f(x,y,z) = sin(xy) + e^{\,2xz}$:

$\frac{\partial f}{\partial x} = cos\,(xy)\,y + e^{\,2xz}\;(2z)$ $\qquad$ $\frac{\partial f}{\partial y} = cos\,(xy)\,x + 0$ $\qquad$ $\frac{\partial f}{\partial z} = 0 + e^{\,2xz}\;(2x)$

Ex: Find the partial derivative $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ of a function $f(x,y,z) = sin^{-1}(xy) + ln\,(xyz)$:

$\frac{\partial f}{\partial x} = \frac{y}{\sqrt{1-(xy)^2}} + \frac{yz}{xyz}$ $\qquad$ $\frac{\partial f}{\partial y} = \frac{x}{\sqrt{1-(xy)^2}} + \frac{xz}{xyz}$ $\qquad$ $\frac{\partial f}{\partial z} = 0 + + \frac{xy}{xyz}$

The gradient

Grad or del operator, or nabla, is an operator used in mathematics, in particular in vector calculus, as a vector differential operator, usually represented by the nabla symbol ∇ .

In the three-dimensional Cartesian coordinate system, the gradient is given by: $\nabla = \frac{\partial}{\partial x}\,i + \frac{\partial}{\partial y}\,j + \frac{\partial}{\partial z}\,k$

Ex: Find the $grad\,f$ $\;(\nabla f)$ where $f = (x\,y\,z)$:

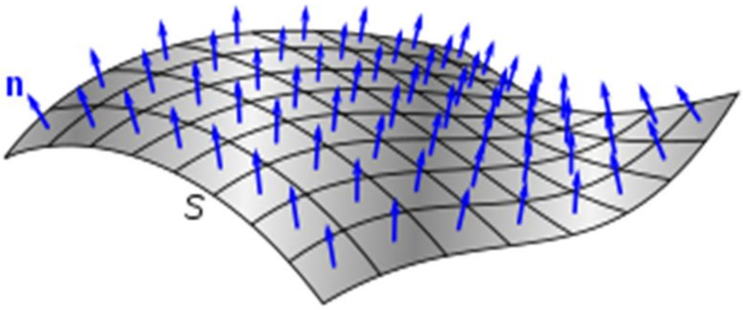
$\nabla f = \frac{\partial f}{\partial x}\,i + \frac{\partial f}{\partial y}\,j + \frac{\partial f}{\partial z}\,k \implies \nabla f = (y\,z)i + (x\,z)\,j + (x\,y)k$

Ex: Find the $grad\,f$ $\;(\nabla f)$ where $f = sin(x^2\,y^2) + e^{(z^2)}$:

$\nabla f = \frac{\partial f}{\partial x}\,i + \frac{\partial f}{\partial y}\,j + \frac{\partial f}{\partial z}\,k$

$\implies \nabla f = [cos(x^2\,y^2)(2x\,y^2) + 0]\,i + [cos(x^2\,y^2)(x^2\;2\,y) + 0]\,j + [0 + e^{(z^2)}\,2z]\,k$

$grad\,f \quad (\nabla f)$



Divergence $div\,F$ or $\nabla . F$

Divergence is a vector operator that operates on a vector field, producing a scalar field giving the quantity of the vector field source at each point.

$$\nabla = \frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k$$

$$\vec{F} = F_x\, i + F_y\, j + F_z\, k$$

$$div\,F \overset{or}{\Longrightarrow} \nabla . F = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right).(F_x\, i + F_y\, j + F_z\, k)$$

EX: Find the divergence of the vector field $\vec{F} = 2xy\, i + yz^2\, j + xz\, k$ at the point $(1, -1, 2)$

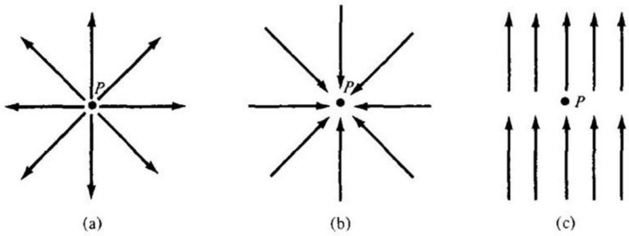
$$\vec{F} = 2xy\, i + yz^2\, j + xz\, k$$

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

$$div\,F \overset{or}{\Longrightarrow} \nabla . F = \left(\frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k\right).(2xy\, i + yz^2\, j + xz\, k)$$

$$\Longrightarrow \nabla . F = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(xz)$$

$$\Longrightarrow \nabla . F = (2y) + (z^2) + (x) \xrightarrow{at\ the\ point\ (1,-1,2)} \nabla . F = 3$$



(a) Positive divergence, (b) negative divergence, (c) zero divergence.

$div\,F > 0$ $\quad div\,F < 0$ $\quad div\,F = 0$

curl

The curl is a vector operator that describes the infinitesimal circulation of a vector field in three-dimensional space.

$$\vec{F} = F_x\, i + F_y\, j + F_z k$$

$$\nabla = \frac{\partial}{\partial x}\, i + \frac{\partial}{\partial y}\, j + \frac{\partial}{\partial z}\, k$$

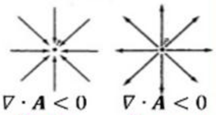
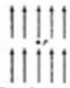
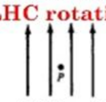
$$\Longrightarrow Curl\,F \overset{Or}{\Longrightarrow} \nabla \times F = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix}$$

EX: Find the $Curl\,F$ where $\vec{F} = (Z^2 y)\, i + (xyz)\, j + (yx^2) k$ at the point $(1, -1, 2)$

$$\Longrightarrow Curl\,F \overset{Or}{\Longrightarrow} \nabla \times F = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (Z^2 y) & (xyz) & (yx^2) \end{bmatrix} \Longrightarrow \nabla \times F = i \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (xyz) & (yx^2) \end{vmatrix} - j \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ (Z^2 y) & (yx^2) \end{vmatrix} + k \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ (Z^2 y) & (xyz) \end{vmatrix}$$

$$\overset{Or}{\Longrightarrow} \nabla \times F = i\left[\frac{\partial}{\partial y}(yx^2) - \frac{\partial}{\partial z}(xyz)\right] - j\left[\frac{\partial}{\partial x}(yx^2) - \frac{\partial}{\partial z}(Z^2 y)\right] + k\left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(Z^2 y)\right]$$

$$\overset{Or}{\Longrightarrow} \nabla \times F = i[(x^2) - (xy)] - j\,[(2yx) - (2z\,y)] + k\,[(yz) - (Z^2)] \xrightarrow{Curl\,F\ at\ the\ point\ (1,-1,2)} \nabla \times F = 2i + 6j - 6k$$

	Gradient	Divergence	Curl
Physical meaning	A vector that gives direction of the maximum rate of change of a quantity i.e. temp	A scalar that measures the magnitude of a source or sink at a given point  $\nabla \cdot A < 0$ Sink i.e. Flux out < flux in; $\nabla \cdot A < 0$ Source i.e. Flux out > flux in  $\nabla \cdot A = 0$ Incompressible Flux out = flux in	A vector operator that describes the rotation/ununiformity of a vector field  $\nabla XA > 0$ RHC rotation  $\nabla XA < 0$ LHC rotation  $\nabla XA = 0$ Irrotational

$Q1$: Find the (∇f) , $\nabla.(\nabla f)$, $\nabla \times (\nabla f)$: $f(x,y,z) = e^{xy} + \cos yz$

$$\nabla f = \frac{\partial}{\partial x}(e^{xy}+\cos yz)\, i + \frac{\partial}{\partial y}(e^{xy}+\cos yz) + j + \frac{\partial}{\partial z}(e^{xy}+\cos yz)\, k$$

$$\nabla f = (ye^{xy}+0)\, i + (x\, e^{xy} - z \sin yz) j + (0 - y \sin yz)\, k \Rightarrow \nabla f = (y\, e^{xy})\, i + (x\, e^{xy} - z \sin yz) j + (y \sin yz) k$$

$$\xrightarrow{div\, of \nabla f} \nabla.(\nabla f) = [(ye^{xy})\, i + (x\, e^{xy} - z \sin yz)\, j - (y \sin yz)\, k\,] . \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)$$

$$\xrightarrow{div\, of \nabla f} \nabla.(\nabla f) = \left(\frac{\partial}{\partial x}(ye^{xy}) + \frac{\partial}{\partial y}(x\, e^{xy} - z \sin yz) + \frac{\partial}{\partial z}(-y \sin yz)\right)$$

$$\xrightarrow{div\ \nabla f} \nabla.(\nabla f) = ((y^2\, e^{xy}) + (x^2\, e^{xy} - z^2 \cos yz) - (y^2 \cos yz))$$

$$\xrightarrow{Curl\ \nabla f} \nabla \times (\nabla f) = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{bmatrix} \xrightarrow{Curl\ \nabla f} \nabla \times (\nabla f) = \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2\, e^{xy}) & (x^2\, e^{xy} - z^2 \cos yz) & (y^2 \cos yz) \end{bmatrix}$$

$Q1$: Find the $grad\, f$ (∇f) where:

$a)\ f(x,y,z) = \sqrt{xyz}$ $\quad b)\ f(x,y,z) = 2^{(xyz)}$ $\quad c)\ f(x,y,z) = x^2\, y + y^2\, z$

$d)\ f(x,y,z) = x^2 \sin y + y^2 \cos z$ $\quad e)\ f(x,y,z) = \sin^{-1}(xy) + \ln(xyz)$

$f)\ f(x,y,z) = \tan(\theta\, y) + \ln(xy) + e^3$ $\quad g)\ f(x,y,\tau) = y \tan(\theta) + x \ln(y) + e^{\pi\tau}$

$h)\ f(x,y,z) = \sin(x^2\, y^2) + e^{(z^2)}$

$Q2$: Find the divergence and the $Curl$ of the vector field for the following:

$a)\ \vec{F} = 2xy\, i + yz^2\, j + xz\, k$ $\quad b)\ \vec{F} = x^2\, z\, i + y\, x^2\, j + y\, z\, k$ $\quad c)\ \vec{F} = xz\, i + y^2 z\, j + xz^2\, k$

$d)\ \vec{F} = 3^{2zx}\, i + \sec(yz^2)\, j + \sin^{-1}(zx)\, k$ $\quad f)\ \vec{F} = \tan^{-1}(xy) i + e^{xy}\, j + \cos yz\, k$

$g)\ \vec{F} = e^{xy}\ i + \cos yz\, j + \tan^{-1}(xy)\, k$