1.1. Finite – Duration and Infinite – Duration Impulse response

It is convenient, however, to subdivide the class of linear – time invariant system into two types, those that have a finite – duration impulse response (FIR) and those that have an infinite – duration impulse response (IIR).

Thus, an FIR system has an impulse response that is zero outside of some finite time interval. The convolution formula for such a system reduces to: -

$$y(n) = \sum_{k=0}^{M-1} x(k)h(n-k)$$

In effect, the system acts as a window that views only the most M input signal samples in forming the output. Thus, we say that an FIR system has a finite memory of length M samples.

In contrast, an IIR linear – time invariant system has an infinite duration impulse response. Its output, based on the convolution formula, is

$$y(n) = \sum_{k=0}^{\infty} x(k)h(n-k)$$

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Difference between FIR and IIR:

Property	FIR Systems	IIR Systems
Impulse Response	Becomes zero after M samples.	Extends indefinitely.
Convolution Formula	Limited to $M - 1$ terms.	Includes an infinite summation.
Memory	Finite (limited to M samples).	Infinite memory (all past inputs).
Design	Easier to design.	More complex, requiring stability analysis.
Stability	Always stable.	Can be unstable without careful design.
Common Applications	Simple digital filters.	Dynamic systems or differential equation modeling.

1.2. Solution of Linear Constant – Coefficient Difference Equations

Given a linear constant – coefficient difference equation as the input – output relationship describing a linear time – invariant system, our objective in this subsection is to determine an explicit expression for the output y(n). The method that is developed is termed the direct method.

Basically, the goal is to determine the output y(n), $n \geq 0$, of the system given a specific input x(n), $n \geq 0$, and a set of initial conditions. The direct solution method assumes that the total solution is the sum of two parts:

$$y(n) = y_h(n) + y_p(n)$$

The part $yh(n)$ is known as the homogeneous or complementary solution, whereas $yp(n)$ is called the particular solution.

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1.2.1.The homogeneous solution of a difference equation.

The problem of solving the linear constant – coefficient difference equation given by

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

By assuming that the input x(n) =0. Thus, we will first obtain the solution to the homogeneous difference equation

$$\sum_{k=0}^{N} a_k\, y(n-k) = 0$$

Basically, we assume that the solution is in the form of an exponential, that is.

$$y_h(n) = \lambda^n$$

Where the subscript *h* on *y(n)* is used to denote the solution to the homogeneous difference equation. If we substitute this assumed solution, we obtain the polynomial equation

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0$$

Or

$$\lambda^{n-N}(\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_{N-1}\lambda + a_N) = 0$$

The polynomial in parentheses is called the characteristic polynomial of the system. In general, it has N roots, which we denote as $\lambda_1, \lambda_2, \ldots \lambda_N$. The roots can be real or complex valued. In practice the coefficients $a_1, a_2, \ldots a_N$ are usually real. For the moment, let us assume that the roots are distinct, that is, there are no multiple – order roots.

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Then the most general solution to the homogeneous difference equation is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \cdots + C_N\lambda_N^n$$

Where $C_1, C_2.....C_N$ are weighting coefficients.

Example1

Determine the homogeneous solution of the system described by the first – order difference equation

$$y(n) + a_1 y(n-1) = x(n)$$

Solution

The assumed solution obtained by setting $x(n) = 0$ is

$$y_h(n) = \lambda^n$$

When we substitute this solution in the above equation with $x(n) = 0$, we obtain

$$\lambda^n + a_1\lambda^{n-1} = 0$$
$$\lambda^{n-1}(\lambda + a_1) = 0$$

λ^{n-1}, which is not possible because $\lambda^{n-1} \neq 0$ for any non-zero λ

And, $\lambda = -a_1$

Therefore, the solution to the homogeneous difference equation is

$$y_h(n) = C\lambda^n = C(-a_1)$$

Submitting n=0 into the homogenous solution:

$$y(0) = C(-a_1)^0$$

Or

$$y_h(0) = C$$

And hence the zero – input response of the system is

$$y_h(n) = (-a_1)^{n+1} y(-1) \qquad n \geq 0$$

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Example2

Determine the zero – input response of the system described by the homogeneous second – order difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

First, we determine the solution to the homogeneous equation. We assume the solution to be the exponential

$$y_h(n) = \lambda^n$$

$$y_h(n) - 3y_h(n-1) - 4y_h(n-2) = 0$$

Upon substitution of this solution, we obtain the characteristic equation

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

$$\lambda^{n-2}(\lambda^2 - 3\lambda - 4) = 0$$

Therefore , λ^{n-2}, which is not possible because $\lambda^{n-2} \neq 0$ for any non-zero λ

And, $\lambda_1 = -1$, $\lambda_2 = 4$

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n$$

$$y_h(n) = C_1(-1)^n + C_2(4)^n$$

Submitting when $n \geq 0$

$$y_h(0) = C_1 + C_2$$
$$y_h(1) = -C_1 + 4C_2$$

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The zero- input response of the system can be obtained from the homogenous solution by evaluating the constants, given the initial conditions *y(-1)* and *y(-2)*. From the difference equation we have

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

$$y(0)=3y(-1)+4y(-2)$$

$$y(1)=3y(0)+4y(-1)$$

$$y(1)=3[3y(-1)-4y(-2)]+4y(-1)$$

$$y(1)=9y(-1)-12y(-2)+4y(-1)$$

$$y(1)=13y(-1)+12y(-2)$$

$$y_h(0) = C_1 + C_2 = 3y(-1)+4y(-2)$$

$$y_h(1) = -C_1 + 4C_2 = 13y(-1)+12y(-2)$$

$$5C_2=16y(-1)+16y(-2)$$

$$C_2=\frac{16}{5}y(-1)+\frac{16}{5}y(-2)$$

$$C_1=\frac{-4}{5}y(-1)+\frac{4}{5}y(-2)$$

$$y_h(n)= C_1\lambda_1^n+C_2\lambda_2^n$$

$$y_h(n)=[\frac{-4}{5}y(-1)+\frac{4}{5}y(-2)](-1)^n+[\frac{16}{5}y(-1)+\frac{16}{5}y(-2)](4)^n$$

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1.2.2.The particular solution of the difference equation.

The particular solution $y_p(n)$ is required to satisfy the difference equation for the specific input signal $x(n)$, $n \geq 0$. In other words, $y_p(n)$ is any solution satisfying

$$\sum_{k=0}^{N} a_k y_p\,(n-k) = \sum_{k=0}^{M} b_k x\,(n-k)$$

To solve above equation, we assume for $y_p(n)$, a form that depends on the form of the input *x(n).*

Example1/ Determine the particular solution of the first order difference equation

$$y(n) + a_1 y(n-1) = x(n) \qquad |a_1| < 1$$

When the input x(n) = u(n).

Solution

Since the input sequence x(n) is a constant for $n \geq 0$, the form of the solution that we assume is also a constant. Hence the assumed solution of the difference equation to the forcing function x(n), called the particular solution of the difference equation, is

$$yp(n) = Ku(n)$$

Where K is a scale factor determined so that the difference equation is

satisfied.

$$Ku(n) + a1Ku(n-1) = u(n)$$

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To determine *K*, we must evaluate this equation for any n ≥ 1. Where none of the terms vanish. Thus

$$K + a_1 K = 1 \qquad K = \frac{1}{1+a_1}$$

Therefore, the particular solution to the difference equation is

$$y_p(n) = \frac{1}{1 + a_1} u(n)$$

In this example, the input *x(n)*, *n≥ 0*, is a constant and the form assumed for the particular solution is also constant. If *x(n)* is an exponential, we would assume that the particular solution is also be a sinusoid. The table below provides the general form of the particular solution for several types of excitations.

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