2: Sample spaces and events … Axioms for probability and their consequences.

2 : فضاء العينة والاحداث... فرضيات الاحتمالات.

Sample Spaces:

A sample space is the set of all possible outcomes of an experiment. We'll denote a sample space as Ω.

Examples:

- Coin flip: Ω = {H, T}   H= Head   T= Tail
- Roll a 6-sided die: Ω = {1, 2, 3, 4, 5, 6}
- Pick a ball from a bucket of red/black balls: Ω = {R, B}

Sets: A set is a collection of unique objects.

Here "objects" can be concrete things (people in class), or abstract things (numbers, colors).

Examples:

A = {3, 8, 31}

B = {apple, pear, orange, grape}

Not a valid set definition: C = {1, 2, 3, 4, 2}

- Order in a set does not matter! {1, 2, 3} = {3, 1, 2} = {1, 3, 2}
- When x is an element of A, we denote this by: $x \in A$.
- If x is not in a set A, we denote this as: $x \notin A$.
- The "empty" or "null" set has no elements: $\emptyset = \{\ \}$

Some Important Sets

- Integers: Z = {. . . , −3, −2, −1, 0, 1, 2, 3, . . .}
- Natural Numbers: N = {0, 1, 2, 3, . . .}
- Real Numbers: R = "any number that can be written in decimal form"

$$5 \in R,\ 17.42 \in R,\ \pi = 3.14159\ldots \in R$$

Building Sets Using Conditionals

- Alternate way to define natural numbers: N = {x ∈ Z : x ≥ 0}
- Set of even integers: {x ∈ Z : x is divisible by 2}
- Rationals: Q = { p/q : p, q ∈ Z, q 6= 0}

Subsets :

Let A is a set and also A is a subset of another set B if every element of A is also an element of B, and we denote this as A ⊆ B.

Examples:

- {1, 9} ⊆ {1, 3, 9, 11}
- Q ⊆ R
- {apple, pear} ⊈ {apple, orange, banana}
- Ø ⊆ A for any set A

Events:

An event is a subset of a sample space.

Examples:

You roll a die and get an even number:

$$\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\}$$

You flip a coin and it comes up "heads":

$$\{H\} \subseteq \{H, T\}$$

Your code takes longer than 5 seconds to run:

$$(5,\infty) \subseteq R$$

Set Operations: Union

The union of two sets A and B, denoted $A \cup B$ is the set of all elements in either A or B (or both).

When A and B are events, $A \cup B$ means that event A or event B happens (or both).

Example:

A = {1, 3, 5} "an odd roll"

B = {1, 2, 3} "a roll of 3 or less"

$A \cup B = \{1, 2, 3, 5\}$

Set Operations: Intersection

The intersection of two sets A and B, denoted $A \cap B$ is the set of all elements in both A and B.

When A and B are events, $A \cap B$ means that both event A and event B happen.

Example:

A = {1, 3, 5} "an odd roll"

B = {1, 2, 3} "a roll of 3 or less"

A ∩ B = {1, 3}

Note: If A ∩ B = ∅, we say A and B are disjoint منفصله.

Set Operations: Complement:

The complement of a set A ⊆ Ω, denoted $\mathbf{A}^c$, is the set of all elements in (Ω = Sample spaces) that are not in A.

When A is an event, $\mathbf{A}^c$ means that the event A does not happen.

Example:

A = {1, 3, 5} "an odd roll"

$\mathbf{A}^c$ = {2, 4, 6} "an even roll"

Set Operations: Difference:

The difference of a set A ⊆ Ω and a set B ⊆ Ω, denoted A − B, is the set of all elements in Ω that are in A and are not in B.

Example:

A = {3, 4, 5, 6}

B = {3, 5}

A − B = {4, 6}

Note: A − B = A ∩ $\mathbf{B}^c$

DeMorgan's Law:

Complement of union or intersection:

$(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

H/W

Check whether the following statements are true or false.

(Hint: you might use Venn diagrams.)

- $A - B \subseteq A$
- $(A - B)^c = A^c \cup B$
- $A \cup B \subseteq B$
- $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

Probability:

A probability function on a finite sample space Ω assigns every event $A \subseteq \Omega$ a number in [0, 1], such that

1. $P(\Omega) = 1$
2. $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$

$P(A)$ is the probability that event A occurs.

Equally Likely Outcomes

The number of elements in a set A is denoted |A|.

If Ω has a finite number of elements, and each is equally likely, then the probability function is given by **$P(A) = |A|/|\Omega|$**

Example: Rolling a 6-sided die

- P({1}) = 1/6
- P({1, 2, 3}) = 1/2

Repeated Experiments

If we do two runs of an experiment with sample space Ω, then we get a new experiment with sample space

$$\Omega \times \Omega = \{(x, y) : x \in \Omega, y \in \Omega\}$$

The element $(x, y) \in \Omega \times \Omega$ is called an **ordered pair**.

Properties:

Order matters: $(1, 2) \neq (2, 1)$ الترتيب مهم

Repeats are possible: $(1, 1) \in N \times N$

More Repeats

Repeating an experiment n times gives the sample space

$\Omega^n = \Omega \times \cdots \times \Omega$ (n times)

$= \{(x_1, x_2, \ldots, x_n) : x_i \in \Omega \text{ for all } i\}$

The element $(x_1, x_2, \ldots, x_n)$ is called an **n-tuple**.

If $|\Omega| = k$, then $|\Omega^n| = k^n$.

Probability Rules

- Complement of an event A:

$$P(A^c) = 1 - P(A)$$

- Union of two overlapping متداخلة events $A \cap B \neq \emptyset$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$