Al-Mustaqbal University

College of Engineering and Technology

Department of Biomedical Engineering

Stage: three

Signal Processing

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Lecture (12): Z- Transform

digital filter (FIR)

Z-Transform

5.1 Definition of Z.T

The z-transform is a very important tool in describing and analyzing digital systems. It also offers the techniques for digital filter design and frequency analysis of digital signals. The z-transform of a ***causal*** sequence $x(n)$, designated by $X(z)$ or $Z(x(n))$, is defined as:

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n)z^{-n}$$
$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \ldots \quad (5.1)$$

Where, z is the complex variable. Here, the summation taken from $n = 0$ to $n = \infty$ is according to the fact that for most situations, the digital signal $x(n)$ is the ***causal*** sequence, that is, $x(n) = 0$ for $n \le 0$. For non-causal system, the summation starts at $n = -\infty$. Thus, the definition in Equation (5.1) is referred to as a **one-sided z-transform** or a **unilateral transform**. The region of convergence is defined based on the particular sequence $x(n)$ being applied. The z-transforms for common sequences are summarized below:

Line No.	$x(n)$, $n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\|z\| > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\|z\| > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\|z\| > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
10	$\cos(an)u(n)$	$\frac{z[z-\cos(a)]}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
11	$a^n \sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2 - [2a\cos(b)]z + a^2}$	$\|z\| > \|a\|$
12	$a^n \cos(bn)u(n)$	$\frac{z[z-a\cos(b)]}{z^2 - [2a\cos(b)]z + a^{-2}}$	$\|z\| > \|a\|$
13	$e^{-an}\sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$
14	$e^{-an}\cos(bn)u(n)$	$\frac{z[z-e^{-a}\cos(b)]}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$
15	$2\|A\|\|P\|^n \cos(n\theta + \phi)u(n)$ where P and A are complex constants defined by $P = \|P\|\angle\theta, A = \|A\|\angle\phi$	$\frac{Az}{z-P} + \frac{A^*z}{z-P^*}$	

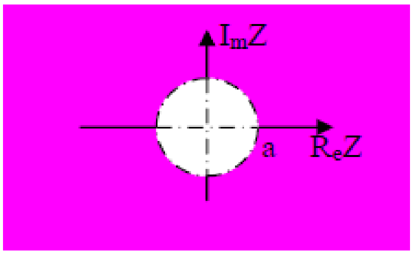
Example (1): Find Z.T including region of convergence of $x(n) = a^n u(n)$

Solution:

$$X(Z) = \sum_{n=0}^{\infty} a^n Z^{-n} = \sum_{n=0}^{\infty} (a Z^{-1})^n = \frac{1}{1 - aZ^{-1}} = \frac{Z}{Z-a} \quad , \quad |a Z^{-1}| < 1$$

Or $|Z| > |a|$

The region of convergence (ROC) is outside the unit circle only.



5.2 Properties of Z.T:

5.2.1 Linearity: The z-transform is a linear transformation, which implies

$$Z\,(a\,x_1(n) \pm b\,x_2(n)) = a\,X_1(Z) \pm b\,X_2(Z) \tag{5.2}$$

Where; a and b are constants. **ROC = ROC1∩ ROC2**

5.2.2 Shift theorem (Delay) (without initial conditions): Given $X(z)$, the z-transform of a sequence $x(n)$, the z-transform of $x(n - m)$, the time-shifted sequence, is given by;

$$Z\{x(n - m)\} = Z^{-m} X(Z) \tag{5.3}$$

5.2.3 Convolution: Given two sequences $x_1(n)$ and $x_2(n)$, their convolution can be determined as follows:

$$x(n) = x_1(n) \otimes x_2 = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) = \sum_{k=-\infty}^{\infty} x_1(n-k)\, x_2(k) \tag{5.4}$$

Where $\otimes$ designates the linear convolution. In z-transform domain, we have

$$X(Z) = X_1(Z) \,.\, X_2(Z) \tag{5.5}$$

Basic Steps:

1. Compute z-Transform of each of the signals to convolve (time domain → z-domain):

$$X_1(z) = Z\,[x_1(n)] \quad , X_2(z) = Z\,[x_2(n)]$$

2. Multiply the two z-Transforms (in z-domain): $X(z) = X_1(z)\, X_2(z)$

3. Find the inverse z-Transform of the product (z-domain → time domain):

$$x(n) = z^{-1}[X(z)]$$

5.2.4 Multiplication by exponential:

$$Z\{a^n\ x(n)\} = X(Z)\Big|_{Z\to\frac{Z}{a}} \qquad (5.6.a)$$

$$Z\{e^{\pm an}x(n)\} = X(Z)\Big|_{Z\to e^{\mp a}Z} \qquad (5.6.b)$$

5.2.5 Initial and final value theorems:

$$\lim_{n\to 0} x(n) = \lim_{Z\to\infty} X(Z) = x(0) \quad \textit{initial value theorem} \qquad (5.7.a)$$

$$\lim_{n\to\infty} x(n) = \lim_{Z\to 1} Z^{-1}\ (Z-1)\ \ X(Z) \quad \textit{final value theorem} \qquad (5.7.b)$$

5.2.6 Multiplication by *n* (Differentiation of *X*(z)):

$$Z\{n\ x(n)\} = -Z\frac{d}{dZ}X(Z) \qquad (5.8)$$

5.3 Inverse of Z.T

$$x(n) = Z^{-1}\{X(Z)\} \qquad (5.9)$$

The inverse z-transform may be obtained by the following methods:

1. Using properties.
2. Partial fraction (P.F) expansion method.
3. Power series expansion (the solution is obtained by applying long division because the denominator can't be analyzed. It is not accurate method compared with the above three methods).

Example (2): Find $x(n)$ using **partial fraction method**, if:

$$X(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})}.$$

Solution:

Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z) = \frac{z^2}{z^2(1-z^{-1})(1-0.5z^{-1})}.$$

$$= \frac{z^2}{(z-1)(z-0.5)}$$

Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}.$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}.$$

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)}\bigg|_{z=1} = 2,$$

$$B = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1.$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}.$$

Multiplying z on both sides gives

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}.$$

$$x(n) = 2u(n) - (0.5)^n u(n).$$

Example (3): Find the inverse transform of $X(z)$ using **partial fraction method**.

$$X(z) = \frac{z}{3z^2-4z+1}$$

Solution:

Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{1}{3z^2-4z+1} = \frac{1}{3(z^2-\frac{4}{3}z+\frac{1}{3})} = \frac{1}{3(z-1)(z-\frac{1}{3})} = \frac{A}{z-1} + \frac{B}{z-\frac{1}{3}}$$

$$A = \frac{X(z)}{z}(z-1)\,|\rightarrow \quad z=1 \quad \rightarrow \quad \frac{1}{3(z-\frac{1}{3})}; \qquad A=1/2$$

$$B = \frac{X(z)}{z}(z-\frac{1}{3})\,|\rightarrow \quad z=\frac{1}{3} \quad \rightarrow \quad \frac{1}{3(z-1)}, \qquad B=-1/2$$

Therefore,

$$\frac{X(z)}{z} = \frac{1/2}{z-1} + \frac{-1/2}{z-\frac{1}{3}}$$

$$X(z) = \frac{\frac{1}{2}z}{z-1} + \frac{\left(-\frac{1}{2}\right)z}{z-\frac{1}{3}}$$

$$\therefore \quad x(n) = \tfrac{1}{2}\,u(n) - \tfrac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$