Matrices

Linearly Dependent & Linearly Independent Functions

If $y_1, y_2, y_3, \ldots, y_m$ are functions of x then $y_1, y_2, y_3, \ldots, y_m$ are linearly dependent if $w(y_1, y_2, y_3, \ldots, y_m) = 0$ where

$$w(y_1, y_2, y_3, \ldots, y_m) = \begin{vmatrix} y_1 & y_2 & \cdots & y_m \\ y_1' & y_2' & \cdots & y_m' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(m-1)} & y_2^{(m-1)} & \cdots & y_m^{(m-1)} \end{vmatrix}$$

if $w(y_1, y_2, y_3, \ldots, y_m) \neq 0$ then $y_1, y_2, y_3, \ldots, y_m$ are linearly independent.

Example

- $y_1 = e^x$, $y_2 = x^2$

$$w(y_1, y_2) = \begin{vmatrix} e^x & x^2 \\ e^x & 2x \end{vmatrix} = 2xe^x - x^2e^x = xe^x(2-x) \neq 0$$

So, y_1 and y_2 are linearly independent.

- $y_1 = 4e^x$, $y_2 = 2e^x$

$$w(y_1, y_2) = \begin{vmatrix} 4e^x & 2e^x \\ 4e^x & 2e^x \end{vmatrix} = 8e^{2x} - 8e^{2x} = 0$$

So, y_1 and y_2 are linearly dependent.

Eigen Values & Eigen Vectors

Let A be an $n \times n$ matrix, a real or complex number λ is called an Eigen value of A if $\det(A-\lambda I)=0$ and with $[A-\lambda I]X=0$ then X is called an Eigen vector with respect to λ where I is the identity matrix.

Example

Find the Eigen values and Eigen vectors of A if

(a) $A=\begin{bmatrix}1 & 2\\ -1 & -2\end{bmatrix}$, (b) $A=\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}$, (c) $A=\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & 1\\ 0 & 0 & -1\end{bmatrix}$

Solution

(a) To find Eigen values, we have $\det(A-\lambda I)=0$ then

$$\begin{vmatrix}1-\lambda & 2\\ -1 & -2-\lambda\end{vmatrix}=0 \Rightarrow (1-\lambda)(-2-\lambda)+2=0 \Rightarrow -2-\lambda+2\lambda+\lambda^2+2=0$$

$$\Rightarrow \lambda^2+\lambda=0 \Rightarrow \lambda(\lambda+1)=0 \Rightarrow \lambda_1=0,\ \lambda_2=-1$$

For $\lambda_1=0 \Rightarrow \begin{bmatrix}1 & 2\\ -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$

$x_1+2x_2=0 \Rightarrow x_1=-2x_2$

Let $x_2=1 \Rightarrow x_1=-2 \Rightarrow$ The Eigen vector for $\lambda_1=0$ is $\begin{bmatrix}-2\\ 1\end{bmatrix}$

For $\lambda_2=-1 \Rightarrow \begin{bmatrix}2 & 2\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$

$2x_1+2x_2=0 \Rightarrow 2x_1=-2x_2 \Rightarrow x_1=-x_2$

Let $x_2=1 \Rightarrow x_1=-1 \Rightarrow$ The Eigen vector for $\lambda_2=-1$ is $\begin{bmatrix}-1\\ 1\end{bmatrix}$

(b) To find Eigen values, we have $\det(A-\lambda I)=0$ then

$$\begin{vmatrix}1-\lambda & 3\\ 2 & 1-\lambda\end{vmatrix}=0 \Rightarrow (1-\lambda)^2-6=0 \Rightarrow 1-2\lambda+\lambda^2-6=0$$

$$\Rightarrow \lambda^2-2\lambda-5=0 \rightarrow$$

$$\Rightarrow \lambda_1=1+\sqrt{6},\ \lambda_2=1-\sqrt{6}$$

To find the Eigen vectors, we have $[A-\lambda I]X=0$

For $\lambda_1=1+\sqrt{6} \Rightarrow \begin{bmatrix}-\sqrt{6} & 3\\ 2 & -\sqrt{6}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=0$

$-\sqrt{6}x_1+3x_2=0 \Rightarrow -\sqrt{6}x_1=-3x_2 \Rightarrow x_1=\frac{3}{\sqrt{6}}x_2$

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Let $x_2 = 1 \;\Rightarrow\; x_1 = \frac{3}{\sqrt{6}} \Rightarrow$ The Eigen vector for $\lambda_1 = 1+\sqrt{6}$ is $\begin{bmatrix} \frac{3}{\sqrt{6}} \\ 1 \end{bmatrix}$

For $\lambda_2 = 1-\sqrt{6} \;\Rightarrow\; \begin{bmatrix} \sqrt{6} & 3 \\ 2 & \sqrt{6} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$\sqrt{6}x_1 + 3x_2 = 0 \;\Rightarrow\; \sqrt{6}x_1 = -3x_2 \;\Rightarrow\; x_1 = \frac{-3}{\sqrt{6}}x_2$$

Let $x_2 = 1 \;\Rightarrow\; x_1 = \frac{-3}{\sqrt{6}} \Rightarrow$ The Eigen vector for $\lambda_2 = 1-\sqrt{6}$ is $\begin{bmatrix} \frac{-3}{\sqrt{6}} \\ 1 \end{bmatrix}$

(c) To find Eigen values, we have $\det(A-\lambda I) = 0$ then

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 0 & -1-\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -(1-\lambda)^2(1+\lambda) = 0$$

$$\Rightarrow \lambda_1 = 1,\; \lambda_2 = 1, \text{ and } \lambda_3 = -1$$

To find the Eigen vectors, we have $[A-\lambda I]X = 0$

For $\lambda_{1,2} = 1 \;\Rightarrow\; \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$-x_2 = 0,\; x_3 = 0,\; -2x_3 = 0$$

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The Eigen vector for $\lambda_{1,2} = 1$ is $\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$

Each choice of α gives us an Eigen vector associated with $\lambda = 1$.

For $\lambda_3 = -1 \Rightarrow \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$$2x_1 - x_2 = 0 \Rightarrow 2x_1 = x_2$$

$$2x_2 + x_3 = 0 \Rightarrow 2x_2 = -x_3$$

$$\text{Let } x_1 = 1 \Rightarrow x_2 = 2 \Rightarrow x_3 = -4$$

The Eigen vector for $\lambda_3 = -1$ is $\begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

Exercises

Find the Eigen values and Eigen vectors of the following matrices

1) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ *Ans.* $1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}; -3, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *Ans.* 0, *any non-zero vector*

3) $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ *Ans.* $5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}; -5, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$