3: Conditional probabilities. Bayes' formula. Independent events.

3 : الاحتمالات الشرطية. صيغة بايز. الأحداث المستقلة.

Conditional probabilities:

Conditional probability questions with solutions are given here for students to practice and understand the concept of conditional probability. By conditional probability, we mean the possibility of happening an event after the occurrence of an event already. For example, a dice is rolled, and the outcome is an odd number, now we have to find the probability of the outcome being a prime number, then,

A ≡ Odd outcome for rolling a dice

B ≡ Outcome being a prime number

Clearly, A already happened thereafter; we have to find the probability of happening B.

Thus, **$P(B|A) = P(A \cap B)/P(A)$,** where $P(A) \neq 0$ and

$P(B|A)$ = probability of occurrence of B given A already happened

$P(A \cap B)$ = probability of occurrence of A and B together

$P(A)$ = probability of occurrence of A.

Conditional Probability of happening of A given B has already occurred is given by

$$P(A|B) = P(A \cap B)/P(B) \text{ or } n(A \cap B)/n(B)$$

- $A \cap B$ = "both events A and B happen"
- $A \cup B$ = "either event A or B (or both) happens"
- A^c = "event A does not happen"

Set Theory Rules: (try drawing Venn diagrams of these)

- Definition of set difference: $A - B = A \cap B^c$ "event A happens, but B does not"
- Associative Law:
$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$
- Commutative Law:
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
- Distributive Law:
$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$
$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$
- DeMorgan's Law:
$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Probability Rules:

- **Inclusion-Exclusion Rule:** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- **Complement Rule:** $P(A^c) = 1 - P(A)$
- **Difference Rule:** $P(A - B) = P(A) - P(A \cap B)$

Exercise: Try deriving these rules from the definition of a probability function. Draw a Venn diagram to convince yourself they work.

Conditional Probability:

$P(A|B)$ = "the probability of event A given that we know B happened"
Formula: $P(A|B) = P(A \cap B)/P(B)$

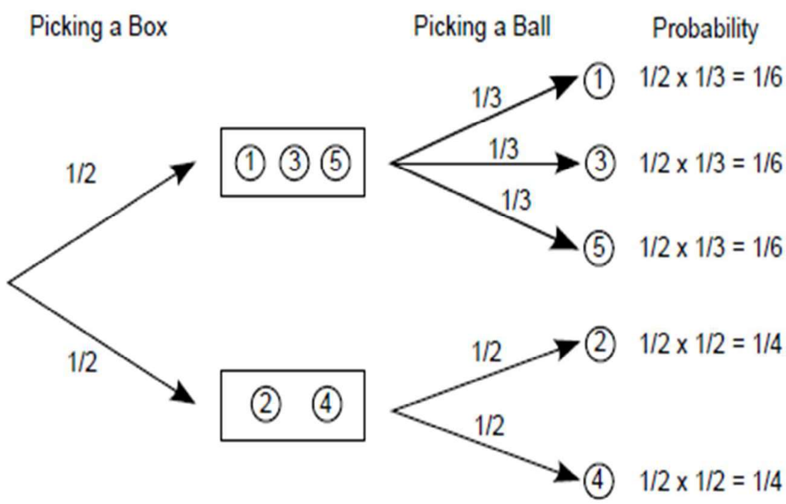
Multiplication Rule:

$$P(A \cap B) = P(A|B)P(B)$$

Tree diagrams to compute "two stage" probabilities (B = first stage, A = second stage):

1. First branch computes probability of first stage: $P(B)$
2. Second branch computes probability of second stage, given the first: $P(A|B)$
3. Multiply probabilities along a path to get final probabilities $P(A \cap B)$

Example: You are given two boxes with balls numbered 1 - 5. One box contains balls 1, 3, 5, and the other contains balls 2 and 4. You first pick a box at random, then pick a ball from that box at random. What is the probability that you pick a 2?



Sampling without replacement:
I have a box with 10 red balls and 10 green balls. I draw 2 balls from the box without replacing them. What is the probability that I get 2 red balls?

Let $R1$ = "first ball red" and $R2$ = "second ball red" and use product rule:

$$P(R1 \cap R2) = P(R1)P(R2|R1) = \frac{1}{2} \times \frac{9}{19} = \frac{9}{38} \approx 0.24$$

If I draw 3 balls without replacement, what is the probability that they are all red?

$$\begin{aligned} P(R1 \cap R2 \cap R3) &= P(R1 \cap R2)P(R3|R1 \cap R2) && \text{Multiplication rule for } (R1 \cap R2) \cap R3 \\ &= P(R1)P(R2|R1)P(R3|R1 \cap R2) && \text{Multiplication rule for } R1 \cap R2 \\ &= \frac{1}{2} \times \frac{9}{19} \times \frac{8}{18} = \frac{18}{171} \approx 0.11 \end{aligned}$$

Conditional probabilities Practice Questions

Question 1:

Ten numbered cards are there from 1 to 15, and two cards a chosen at random such that the sum of the numbers on both the cards is even. Find the probability that the chosen cards are odd-numbered.

Solution:

Let, A ≡ event of selecting two odd-numbered cards

B ≡ event of selecting cards whose sum is even.

Then,

n(B) = number of ways of choosing two numbers whose sum is even

$= {}^{8}C_2 + {}^{7}C_2.$

n(A ∩ B) = number of ways of choosing odd-numbered cards such that their sum is even.

$= {}^{8}C_2.$

Now, P(A|B) = P(A ∩ B)/P(B) = n(A ∩ B)/n(B)

$= {}^{8}C_2 / ({}^{8}C_2 + {}^{7}C_2) = 4/7.$

Question 2:

Let E and F are events of a experiment such that P(E) = 3/10 P(F) = ½ and P(F|E) = ⅖. Find the value of (i) P(E ∩ F) (ii) P(E|F) (iii) P(E U F)

Solution:

We know that P(A|B) = P(A ∩ B)/P(B) ⇒ P(A ∩ B) = P(A|B).P(B)

∴ P(E ∩ F) = P(F|E).P(E) =⅖ × 3/10 = 3/25

(ii) $P(E|F) = P(E \cap F)/P(F) = (3/25) \div (1/2) = 6/25$

(iii) $P(E \cup F) = P(E) + P(F) - P(E \cap F) = 3/10 + 1/2 - 3/25 = 17/25$.

Some properties of conditional probability are:

- Let A and B be the events of a sample space S of an experiment. Then $P(S|B) = P(S|A) = 1$
- Let A and B be the events of a sample space S of an experiment and let E be an event such that $P(E) \neq 0$. Then, $P[(A \cup B)|E] = P(A|E) + P(B|E) - P[(A \cap B)|E]$
- $P[(\text{not } A)|B] = 1 - P(A|B)$
- If A and B are disjoint events then $P(A \cap B) = 0$

Question 3:

The probability of a student passing in science is $4/5$ and the of the student passing in both science and maths is $1/2$. What is the probability of that student passing in maths knowing that he passed in science?

Solution:

Let A ≡ event of passing in science

B ≡ event of passing in maths

Given, $P(B) = 4/5$ and $P(A \cap B) = 1/2$

Then, probability of passing maths after passing in science $= P(B|A) = P(A \cap B)/P(A)$

$= 1/2 \div 4/5 = 5/8$

∴ the probability of passing in maths is $5/8$.

Question 4:

In a survey among few people, 60% read Hindi newspaper, 40% read English newspaper and 20% read both. If a person is chosen at random and if he already reads English newspaper find the probability that he also reads Hindi newspaper.

Solution:

Let there be 100 people in the survey, then

Number of people read Hindi newspaper = n(A) = 60

Number of people read English newspaper = n(B) = 40

Number of people read both = $n(A \cap B) = 20$

Probability of the person reading Hindi newspaper when he already reads English newspaper is given by –

$P(A|B) = n(A \cap B)/n(B) = 20/40 = ½$.

Question 5:

A fair coin is tossed twice such that E: event of having both head and tail, and F: event of having atmost one tail. Find P(E), P(F) and P(E|F)

Solution:

The sample space S = { HH, HT, TH, TT}

E = {HT, TH}

F = {HH, HT, TH}

$E \cap F$ = {HT, TH}

P(E) = 2/4 = ½

P(F) = ¾

$P(E \cap F) = 2/4 = \frac{1}{2}$

$P(E|F) = P(E \cap F)/P(F) = \frac{1}{2} \div \frac{3}{4} = \frac{2}{3}$.

Question 6:

In a class, 40% of the students like Mathematics and 25% of students like Physics and 15% like both the subjects. One student select at random, find the probability that he likes Physics if it is known that he likes Mathematics.

Solution:

Let there be 100 students, then,

Number of students like Mathematics = $n(A) = 40$

Number of students like Physics = $n(B) = 25$

Number of students like both Mathematics and Physics = $n(A \cap B) = 15$

Now, the probability that the student likes Physics if it is known that he likes Mathematics is given by –

$P(B|A) = n(A \cap B)/n(A) = 15/40 = \frac{3}{8}$.

Question 7:

Two dice are rolled, if it is known that atleast one of the dice always shows 4, find the probability that the numbers appeared on the dice have a sum 8.

Solution:

Let,

A: one of the outcomes is always 4

B: sum of the outcomes is 8

Then, A = {(1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6)}

B{(4, 4), (5, 3), (3, 5), (6, 2), (2, 6)}

$n(A) = 11, n(B) = 5, n(A \cap B) = 1$

$P(B|A) = n(A \cap B)/n(A) = 1/11$.

Question 8:

A bag contains 3 red and 7 black balls. Two balls are drawn at randon without replacement. If the second ball is red, what is the probability that the first ball is also red?

Solution:

Let A: event of selecting a red ball in first draw

B: event of selecting a red ball in second draw

$P(A \cap B)$ = P(selecting both red balls) = $3/10 \times 2/9 = 1/15$

P(B) = P(selecting a red ball in second draw) = P(red ball and rad ball or black ball and red ball)

= P(red ball and red ball) + P(black ball and red ball)

$= 3/10 \times 2/9 + 7/10 \times 3/9 = 3/10$

$\therefore P(A|B) = P(A \cap B)/P(B) = 1/15 \div 3/10 = 2/9$.

Question 9:

If a family has two children, what is the conditional probability that both are girls if there is atleast one girl?

Solution:

Let A: both being girls

B: Atleast one girl

n(A) = 1

n(B) = 3

n(A ∩ B) = 1

P(A|B) = n(A ∩ B)/n(B) = ⅓.

Question 10:

A dice and a coin are tossed simultaneously. Find the probability of obtaining a 6, given that a head came up.

Solution:

Let A: six coming with a heads

B: coin shows a head

A ={(6, H))

B = {(1, H), (2, H), (3, H), (4, H), (5, H), (6, H)}

n(A ∩ B) = 1 and n(B) = 6

∴ Probability of getting a six when there is a head is given by –

P(A|B) = n(A ∩ B)/n(B) = ⅙.

11: If P(A) = 0.3, P(B) = 0.7 and P(A∩B) = 0.1 then find P(A/B) and P(B / A).

Given: *P(A) = 0.3, P(B) = 0.7 and P(A∩B) = 0.1*
Thus, P(A / B) = P(A∩B) / P(B)
⇒ P(A / B) = 0.1 / 0.7
⇒ P(A / B) = 1 / 7
and P(B / A) = P(A∩B) / P(A)

⇒ P(B / A) = 0.1 / 0.3
⇒ P(B / A) = 1 / 3

12: If P(A) = 0.2, P(B/A) = 0.8 and P(A∪B) = 0.3 then find P(B).

Given: *P(A) = 0.2, P(B/A) = 0.8 and P(A∪B) = 0.3*
We know, P(A∩B) = P(B / A) × P(A)
⇒ P(A∩B) = 0.2 × 0.8
⇒ P(A∩B) = 0.16
and P(A∩B) = P(A) + P(B) - P(A∪B)
⇒ P(B) = P(A∩B) - P(A) + P(A∪B)
⇒ P(B) = 0.16 - 0.2 + 0.3
⇒ P(B) = 0.26

13: A dice is rolled. If X = {1, 2, 6}, Y = {2, 4} and Z = {2, 4, 6} then, find P(X/Y), P(X/Z).

Given: X = {1, 2, 6}, Y = {2, 4} and Z = {2, 4, 6}
Thus, n(X) = 3, n(Y) = 2, n(Z) = 3
and n(X ∩ Y) = 1, n(X ∩ Z) = 2
Using Formula P(X / Y) = n(X ∩ Y) / n(Y)
⇒ ***P(X / Y) = 1 / 2***
and P(X / Z) = n(X ∩ Z) / n(Z)
Thus, P(X / Z) = 2 / 3

14: A coin is tossed 3 times. Find the conditional probability P(C/D) where C = Head on first Toss and D = Tail on second toss

Sample space of three coin toss is
S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
n(C) = 4, n(D) = 4
Thus, P(C) = 4/ 8 = 1/2
and P(D) = 4/ 8 = 1/2
Using Formula, P(C ∩ D) = 2 / 8 = 1 /4
⇒ P(C /D) = P(C ∩ D) / P(D)
⇒ P(C /D) = (1 / 4) / (1/2)
Thus, P(C / D) = 1 /2

15: Given that P(A/B) = 3/ 7 then, find the conditional probability $P(A_c /B)$.

By the property of conditional probability
$P(A_c / B) = 1 - P(A/B)$
$\Rightarrow P(A_c / B) = 1 - 3/7$
Thus, $P(A_c / B) = 4/7$

16: Evaluate P(X∪Y), if 3P(X) = P(Y) = 2/5 and P(X/Y) = 4/5.

Given: $3P(X) = 2/5$
$\Rightarrow P(X) = 2/15$
and $P(Y) = 2/5$
Using Formula, $P(X \cap Y) = P(X / Y) \times P(Y)$
$\Rightarrow P(X \cap Y) = 4/5 \times 2/5$
$\Rightarrow P(X \cap Y) = 8/25$
Now, $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$
$\Rightarrow P(X \cup Y) = 2/15 + 2/5 - 8/25$
$\Rightarrow P(X \cup Y) = 16/75$

17: If P(A) = 5/8, P(B) = 1/4 and P(A /B) = 3 / 8. Find $P(A_c/ B_c)$.

As we know, $P(B_c) = 1 - P(B)$
$\Rightarrow P(B_c) = 1 - 1/4$
Thus, $P(B_c) = 3/4$
Using Fromula, $P(A \cap B) = P(A / B) \times P(B)$
$\Rightarrow P(A \cap B) = 3/8 \times 1/4$
$\Rightarrow P(A \cap B) = 3/32$
Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\Rightarrow P(A \cup B) = 5/8 + 1/4 -3/32$
$\Rightarrow P(A \cup B) =25/32$
Again, $P(A_c \cap B_c) = 1 - P(A \cup B)$
$\Rightarrow P(A_c \cap B_c)= 1 - 25/32$
$\Rightarrow P(A_c \cap B_c)= 7/32$
$\Rightarrow P(A_c/ B_c) = P(A_c \cap B_c) /P(B_c)$
$\Rightarrow P(A_c/ B_c) = (7/32)/(3/4)$
Thus, $P(A_c/ B_c) = 7/24$

18: In a school there are 100 students out of which 40 are boys. It is known that out of 40, 20% of boys study in class 11. What is the probability P(A/B)

where A: student chosen randomly studies in class 11 and B: the chosen student is a boy.

n(A∩B) = 20% × 40 = 8
n(B) = 40
Using Formula, P(A / B) = n(A∩B) / n(B)
⇒ P(A / B) = 8 / 40
⇒ P(A / B) = 1/5

19: A coin is tossed 2 times. Find P(Y/X) if X = at least one head and Y = no tail.

Sample Space for 2 coin toss is:
S = {HH, HT, TH, TT}
X = {HH, HT, TH} and Y = {HH}
Thus, P(X) = 3/4,
P(Y) = 1/4, and
P(X∩Y) = 1/4
Thus, P(Y/X) = (1/4) / (3/4)
⇒ P(Y/X) = 1 / 3

20: A coin is tossed 3 times. Find (X/Y) if X = at most one head and Y = at most 2 tail.

Sample Space for 3 coin toss is:
S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
P(X) = 4 / 8 = 1/2
P(Y) = 7/8
P(X∩Y) = 3/8
Now, P(X / Y) = P(X∩Y) / P(Y)
⇒ P(X / Y) = (3/8) / (7/8)
⇒ P(X / Y) = 3/7

Conditional probabilities. Bayes' formula

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Bayes Theorem Statement

Let $E_1, E_2, \ldots, E_n$ be a set of events associated with a sample space S, where all the events $E_1, E_2, \ldots, E_n$ have nonzero probability of occurrence and they form a partition of S. Let A be any event associated with S, then according to Bayes theorem,

$$P(E_i\,|\,A) = \frac{P(E_i)P(A\,|\,E_i)}{\sum_{k=1}^{n} P(E_k)P(A|E_k)}$$

for any k = 1, 2, 3, ..., n

Bayes Theorem Proof

According to the conditional probability formula,

$$P(E_i\,|\,A) = \frac{P(E_i \cap A)}{P(A)} \ldots (1)$$

Using the multiplication rule of probability,

$$P(E_i \cap A) = P(E_i)P(A\,|\,E_i) \ldots (2)$$

Using total probability theorem,

$$P(A) = \sum_{k=1}^{n} P(E_k)P(A|E_k) \ldots (3)$$

Putting the values from equations (2) and (3) in equation 1, we get

$$P(E_i\,|\,A) = \frac{P(E_i)P(A\,|\,E_i)}{\sum_{k=1}^{n} P(E_k)P(A|E_k)}$$