



Polar Coordinates

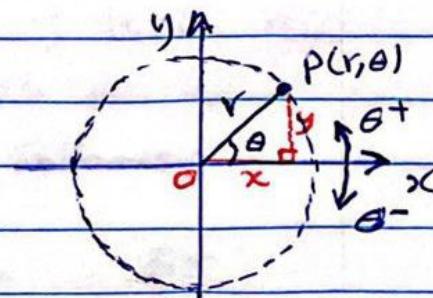
The Polar Coordinates can be defined as a pair of r & θ " (r, θ) ", where r is the distance of a point from the origin and θ is the angle between the line segment from the origin to the point and the x-axis.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1} \frac{y}{x}$$



The above set of Four equations are using to Convert the polar to Cartesian (rectangular) and Vice versa.

تستخدم هذه المعادلات لتحويل الإحداثيات القطبية إلى الإحداثيات الديكارتية (المربعة) والعكس.

$$(r, \theta) \Leftrightarrow (x, y)$$



Ex I Convert $(-1, -1)$ into polar coordinates?

Solution

$$(x, y) \rightarrow (r, \theta)$$

$$x = -1, y = -1$$

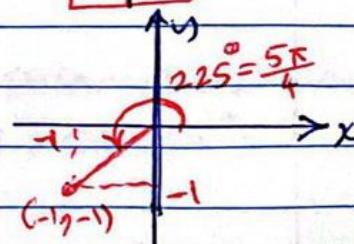
$$r = \sqrt{x^2 + y^2} \Rightarrow r = \sqrt{(-1)^2 + (-1)^2} = \sqrt{1+1} = \boxed{\sqrt{2}}$$

$$\theta = \tan^{-1} \frac{y}{x} \Rightarrow \theta = \tan^{-1} \frac{-1}{-1} = \tan^{-1} 1 = \boxed{45^\circ}$$

Since the point $(-1, -1)$ is in quadrant ③
 $\therefore$ the angle 45 is not correct & need
to be corrected as follows;

$$\theta = \frac{\pi}{4} + \pi = 45 + 180 = \boxed{\frac{5\pi}{4}} = \boxed{225^\circ}$$

$$\therefore (-1, -1) = (\sqrt{2}, \frac{5\pi}{4})$$



Ex II Convert $(4, 60^\circ)$ into rectangular coordinates

Solution

$$(r, \theta) \rightarrow (x, y) ; r = 4, \theta = 60^\circ$$

$$x = r \cos \theta \Rightarrow x = 4 \cos 60 = 4 \times \frac{1}{2} = \boxed{2}$$

$$y = r \sin \theta \Rightarrow y = 4 \sin 60 = 4 \times \frac{\sqrt{3}}{2} = \boxed{2\sqrt{3}}$$

$$\therefore (4, 60^\circ) = (2, 2\sqrt{3})$$



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Ex) Convert $(6, \frac{5\pi}{6})$ into Cartesian Coord?

Solution

$$(r, \theta) \rightarrow (x, y); \quad r = 6, \quad \theta = \frac{5\pi}{6}$$

لنفرض أن (r, θ) هي الإحداثيات القطبية و (x, y) هي الإحداثيات الديكارتية

$$\theta = \frac{5\pi}{6} \times \frac{180}{\pi} = \frac{5 \times 180}{6} = 150^\circ$$

$$x = r \cos \theta \Rightarrow x = 6 \times \cos 150 = 6 \times \left(-\frac{\sqrt{3}}{2}\right) = -3\sqrt{3}$$

$$y = r \sin \theta \Rightarrow y = 6 \times \sin 150 = 6 \times \left(\frac{1}{2}\right) = 3$$

$$\therefore (6, \frac{5\pi}{6}) = (-3\sqrt{3}, 3)$$

Question For Discussion :-

Convert $(2, -4)$ into Polar Coordinates (r, θ) ?
Ans: $(2\sqrt{5}, 296.56^\circ)$

Convert $(-5, 5\sqrt{3})$ into (r, θ) ? Ans: $(10, 120^\circ)$

Convert $(7, 140^\circ)$ into (x, y) ? Ans: $(-5.362, 4.5)$

Convert $(1.5, 300^\circ)$ into (x, y) ? Ans: $(0.75, -1.3)$



Ex) Convert $2x - 5x^3 = 1 + xy$ into polar coordinates?

Solution

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\theta = \tan^{-1} \frac{y}{x}$$

$\therefore$ in polar coordinates $x = r \cos \theta$, $y = r \sin \theta$

$$2r \cos \theta - 5(r \cos \theta)^3 = 1 + (r \cos \theta)(r \sin \theta)$$

$$2r \cos \theta - 5r^3 \cos^3 \theta = 1 + r^2 \sin \theta \cos \theta \quad \text{Ans}$$

Ex) Convert the polar equation $4r \cos \theta + r \sin \theta = 8$ into Cartesian coordinates equation that express y in term of x ?

Solution

$$x = r \cos \theta$$

$$r \cos \theta = x \Rightarrow r = \frac{x}{\cos \theta}$$

$$y = r \sin \theta$$

$$r \sin \theta = y \Rightarrow r = \frac{y}{\sin \theta}$$

$$4(r \cos \theta) + r \sin \theta = 8$$

$$4x + y = 8$$

$$\therefore y = 8 - 4x \quad \text{Ans}$$



Ex1 Convert $r = -8 \cos \theta$ into rectangular coordinates?

Solution

$$x = r \cos \theta \quad ; \quad y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2} \implies r^2 = x^2 + y^2$$

هنا نحتاج الى ضرب الطرفين بالمعادلة r لتصبح المعادلة كالتالي:

$$r^2 = -8 r \cos \theta$$

$$\therefore \boxed{x^2 + y^2 = -8x} \quad \underline{\text{Ans}}$$

كذلك يمكننا ان نعوّض عن $\cos \theta$ بـ $\frac{x}{r}$ وهو كالتالي:
تأنيث لكل كافي التالي:-

$$r = -8 \times \frac{x}{r} \implies r^2 = -8x$$

هنا نعوّض عن r^2 بالمعادلة $(x^2 + y^2)$

$$\therefore \boxed{x^2 + y^2 = -8x} \quad \underline{\text{Ans}}$$

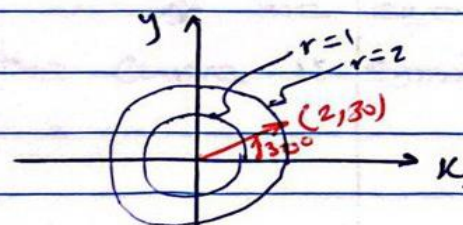


Graph of Polar Functions

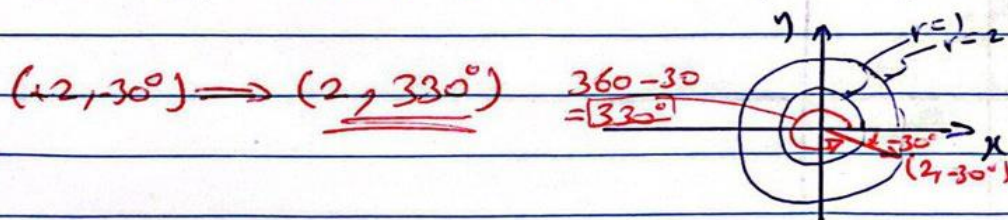
رسم الدوال القطبية

في هذا الدرس سوف ندرس الدوال القطبية لنرى كيف نرسمها
كيفية تحويلها من القطبية إلى الديكارتي:

Ex) Represent the point $(2, 30^\circ)$ onto polar plane?



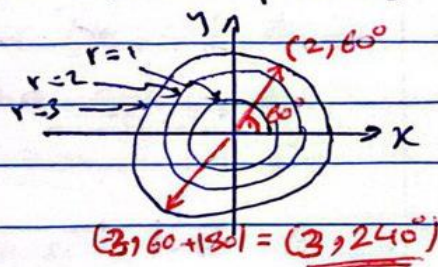
Ex) Rep- $(2, -30^\circ)$ onto polar plane?



Ex) Rep- $(-3, 60^\circ)$ onto polar plane?

نلاحظ

في الدالة $(-3, 60^\circ)$ حيث أن r سالبة
الزاوية تكون باتجاه عقارب الساعة
من الزاوية تكون زاوية مكافئة
لا اتجاه r فتصبح الزاوية
تكون $240^\circ = (60 + 180)$
أي $(3, 240^\circ)$



CS

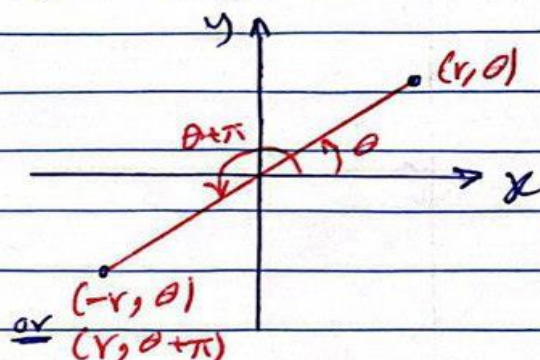
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 $(-3, 60^\circ)$ Equivalent to $(3, 240^\circ)$



③ About the origin (Pole) $\Rightarrow$ also

If r is replaced by $(-r)$ & θ is replaced by $(\pi + \theta)$ the eqⁿ does not change, then the curve is symmetric about the origin

Ex) $\Rightarrow r^2 = (-r)^2$



Ex) Show that the eq^s below are equivalents?

① $r = 1 - \sin(\theta - \frac{\pi}{3})$

② $r = 1 + \cos(\theta + \frac{\pi}{6})$

③ $r = \cos(\theta + \frac{\pi}{6}) - 1$

Solution

① $r = 1 - \sin(\theta - \frac{\pi}{3})$

$$= 1 - (\sin\theta \cos\frac{\pi}{3} - \cos\theta \sin\frac{\pi}{3})$$

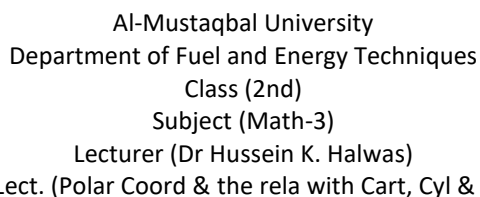
$$= 1 - (\frac{1}{2} \sin\theta - \frac{\sqrt{3}}{2} \cos\theta) \quad \checkmark$$

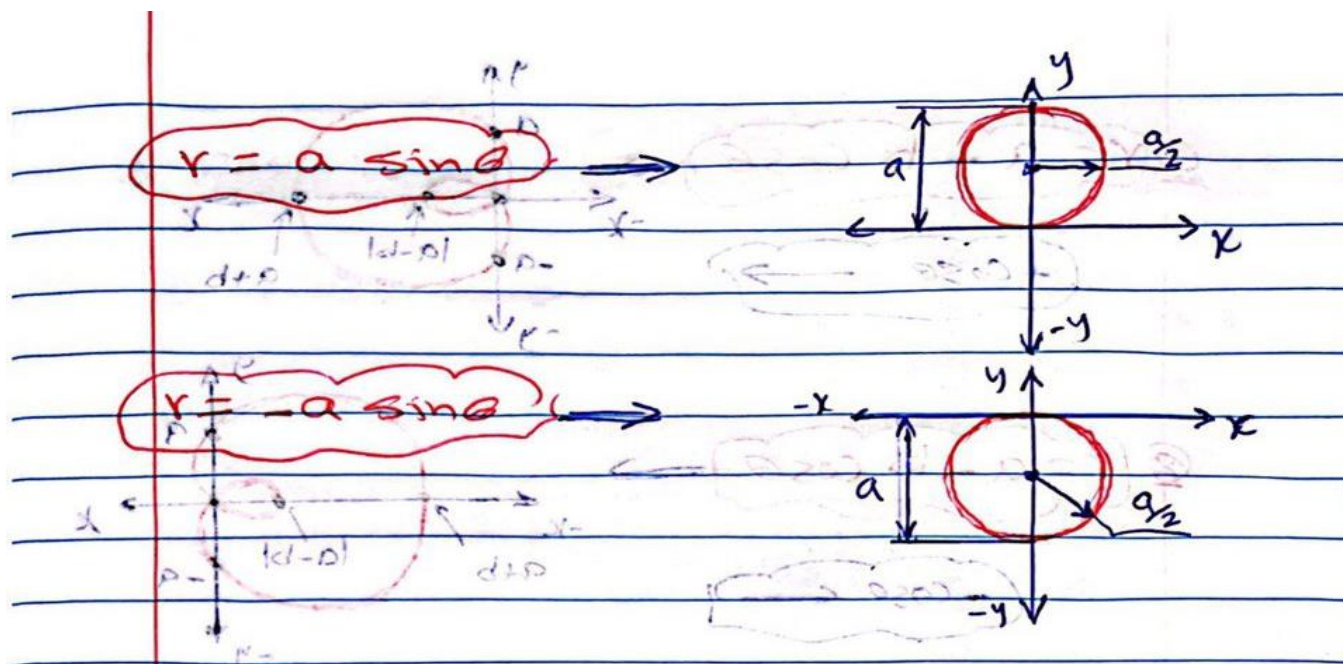
② $r = 1 + \cos(\theta + \frac{\pi}{6})$

$$= 1 + (\cos\theta \cos\frac{\pi}{6} - \sin\theta \sin\frac{\pi}{6})$$

$$= 1 + (\frac{\sqrt{3}}{2} \cos\theta - \frac{1}{2} \sin\theta)$$

$$= 1 - (\frac{1}{2} \sin\theta + \frac{\sqrt{3}}{2} \cos\theta) \quad \checkmark$$





⑥ Limasons

The general equations of limasons are

$$r = a \pm b \cos \theta$$

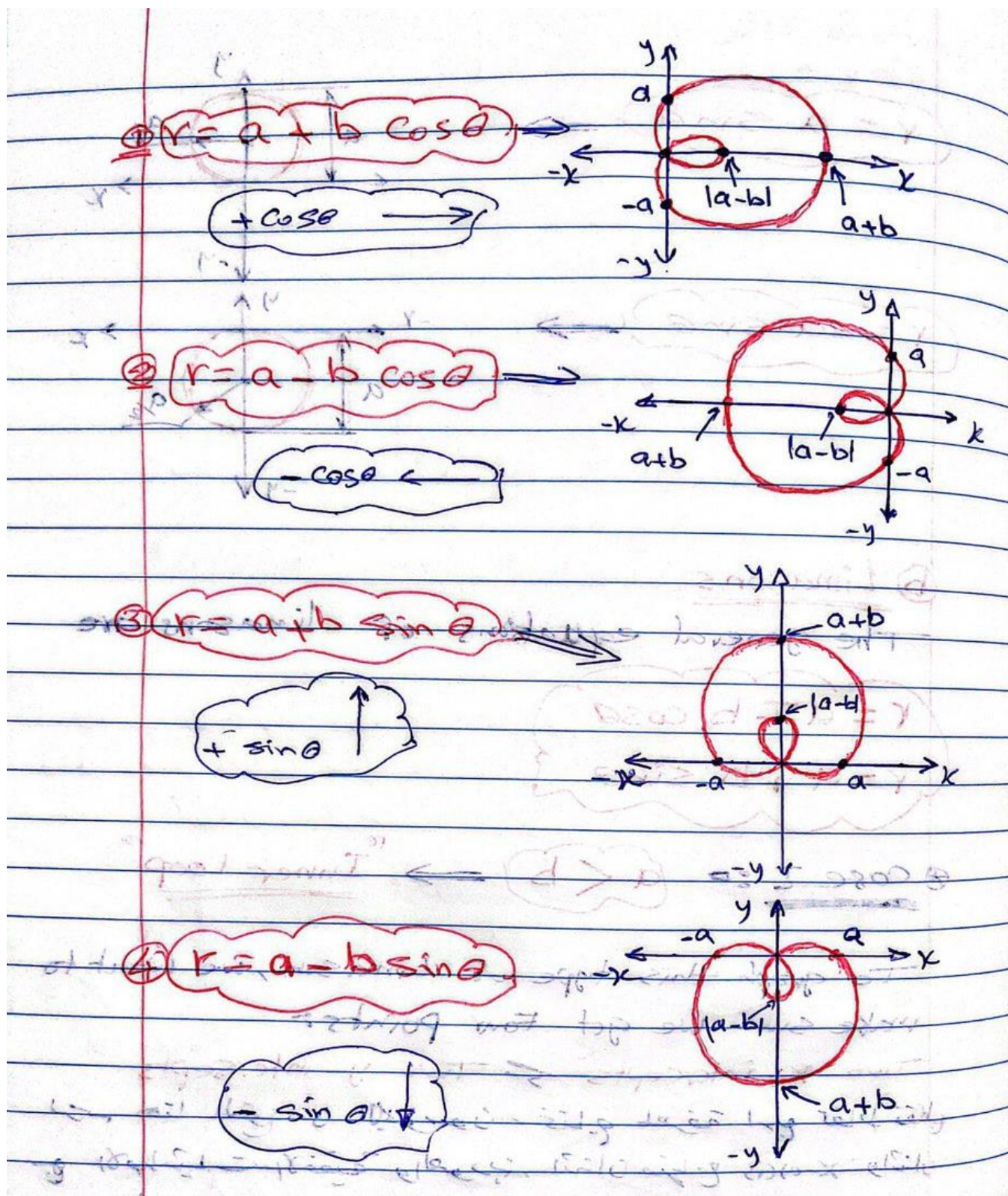
$$r = a \pm b \sin \theta$$

⊗ Case I : $a < b \Rightarrow$ "Inner Loop"

To graph this type of Limason, we want to make sure we get four points:

Two x-intercepts & Two y-intercepts

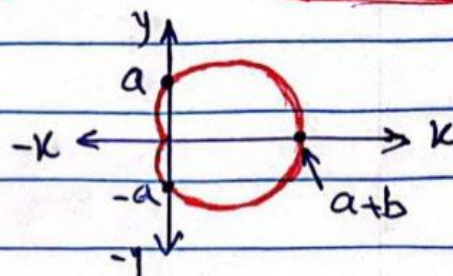
لرسم هذا النوع من الليماسون - نحتاج طريقة اربع نقاط تقاطع
مع المحاور الافقية والعمودية، اثنين مع x -axis واثنين
مع y -axis. الطريقة الثانية - نقرأ تقاطع الرسم



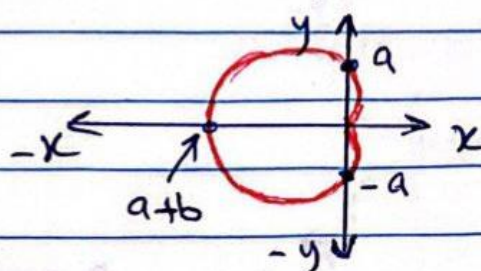


* Case II : $a = b$ → "Cardioid"

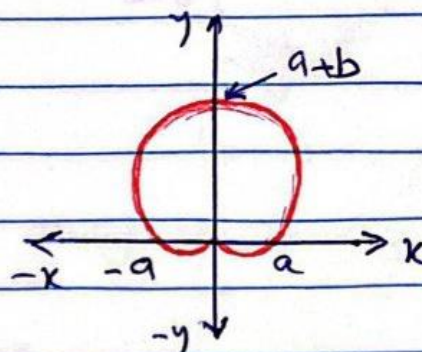
① $+ \cos \theta$ →



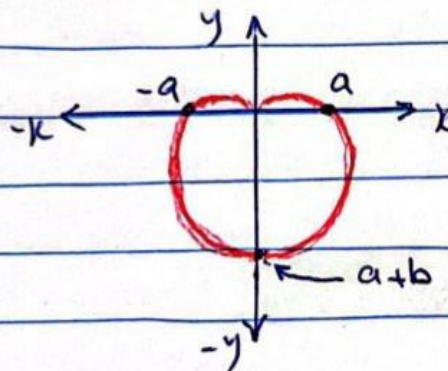
② $- \cos \theta$ →



③ $+ \sin \theta$ →



④ $- \sin \theta$ →





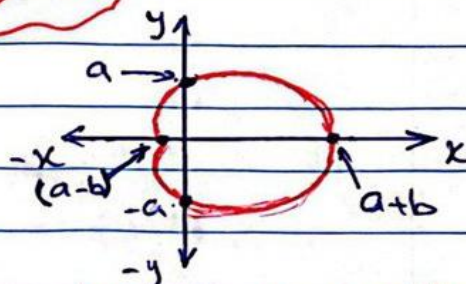
Case III :-

$$a > b$$

a -

$$b < a < 2b$$

∴ $\cos \theta$ is ≥ 0 & ≤ 1 ∴ θ is in 1st & 4th quadrant

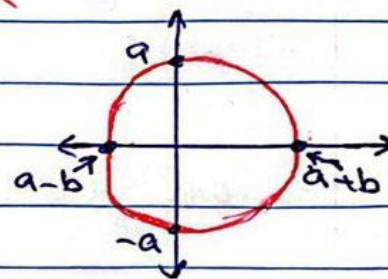


(Dimple Cardioid)

b -

$$a \geq 2b$$

∴ $\cos \theta$ is ≥ 0 & ≤ 1 ∴ θ is in 1st & 4th quadrant



(No Dimple)

Examples :-

① Graph $r = 4 \cos \theta$

Soln

The shape is circle, to check about which axis is symmetric we do the follows:

① put $r = -r \Rightarrow -r = 4 \cos \theta$ (not symm. about the origin)

② put $\theta = \pi - \theta \Rightarrow r = 4 \cos(\pi - \theta)$
 $= 4(\cos \pi \cos \theta + \sin \pi \sin \theta)$

$$r = -4 \cos \theta$$

(not symm. about y-axis)



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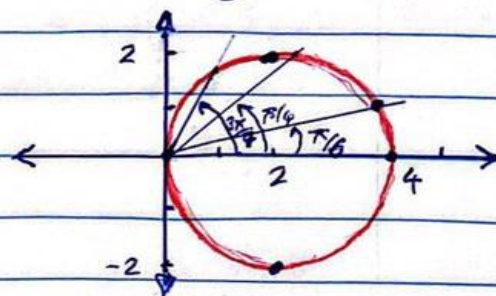


③ Put $\theta = -\theta \Rightarrow r = 4 \cos(-\theta) = \underline{\underline{4 \cos \theta}}$

(Symmetry about the x-axis)

هذا هو شكل المعادلة القطبية $r = 4 \cos \theta$ ، وهو دائرة تقع على x-axis بقطر 4.

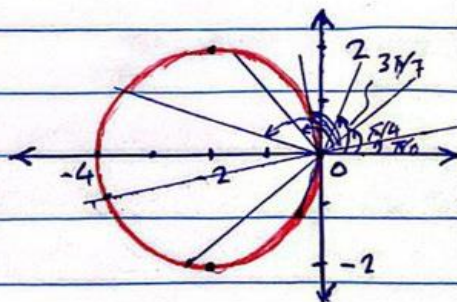
r	θ
4	0
3.46	$\frac{\pi}{6}$
2.83	$\frac{\pi}{4}$
0	$\frac{\pi}{2}$
-2.83	$\frac{3\pi}{4}$
-4	π



② Sketch $r = -4 \cos \theta$

هنا نرى نفس الخطوات في المثال ①، لكن مع إشارة سالبة. هذا يعني أن الدائرة ستكون متناظرة حول x-axis، ولكن تقع على الجانب السلبي. هذا هو شكل المعادلة القطبية $r = -4 \cos \theta$ ، وهو دائرة تقع على x-axis بقطر 4.

r	θ
-4	0
-3.46	$\frac{\pi}{6}$
-2.83	$\frac{\pi}{4}$
0	$\frac{\pi}{2}$
2.83	$\frac{3\pi}{4}$
4	π





③ Graph $r = 2 \sin \theta$

الدالة هي دالة جيبية، ونعطي رسم لكل متناظر حول

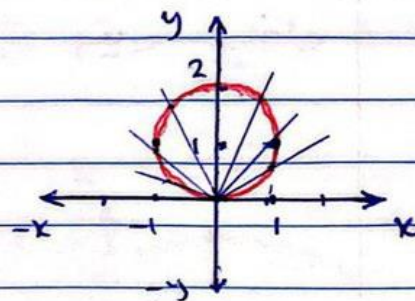
y-axis، ونلاحظ أن ذلك يعطينا مكان $\theta = \pi - \theta$

$$\therefore \text{put } \theta = \pi - \theta \rightarrow r = 2 \sin(\pi - \theta) = 2 \sin \theta$$

من خلال المعادلة العامة نلاحظ أن $r = 2 \sin \theta$ يعطينا

دائرة متناظرة حول y-axis، ويقطرها 2 و -2

r	θ
0	0
1	$\pi/6$
1.414	$\pi/4$
2	$\pi/2$
1.414	$3\pi/4$
0	π

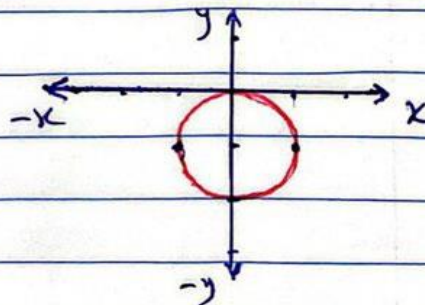


④ Sketch $r = -2 \sin \theta$

الصورة تمامًا نفس الموجور في المثال السابق لكن يقع

على y-axis، والـ 2 كما أنه يقطرها 2 و -2

r	θ
0	0
-1	$\pi/6$
-1.414	$\pi/4$
-2	$\pi/2$
-1.414	$3\pi/4$
0	π





⑤ Graph $r = 3 + 5 \cos \theta$

Soln (compare to $r = a + b \cos \theta$)

It's a type of a limacon's graphs! and as $a = 3$, $b = 5 \Rightarrow a < b \Rightarrow$ "Inner loop" and as $+\cos \theta$ has +ve sign, then it is located on the x-axis and symmetric on it.

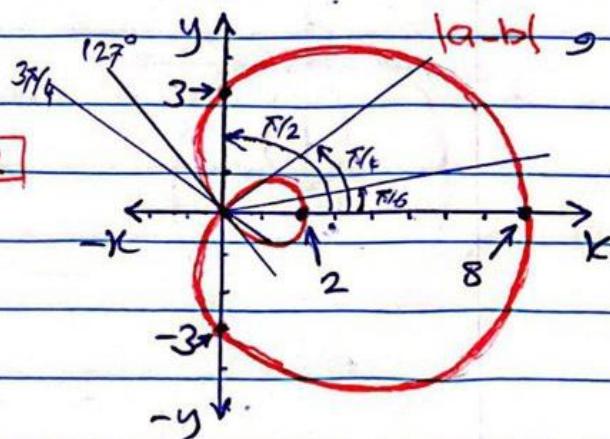
هذا النوع من الدوال يسمى بالدالة القطبية ويكون شكلها على المحور x ويكون متماثل على المحور x.

لأن $a < b$ فسيكون الشكل على شكل حلقة داخلية ويكون متماثل على المحور x.

$$a = 3$$

$$a + b = 3 + 5 = 8$$

$$|a - b| = |3 - 5| = |-2| = 2$$



r	θ
8	0

7.33	$\pi/6$
------	---------

6.53	$\pi/4$
------	---------

3	$\pi/2$
---	---------

0	$3\pi/4$ 127
---	--------------

-0.535	$3\pi/4$
--------	----------

-2	π
----	-------



⑦ Graph $r = 3 - 7 \cos \theta$

Sol

compare to $r = a + b \cos \theta \rightarrow$ "limasans"

$a = 3, b = 7 \rightarrow a < b \rightarrow$ "Inner loop"

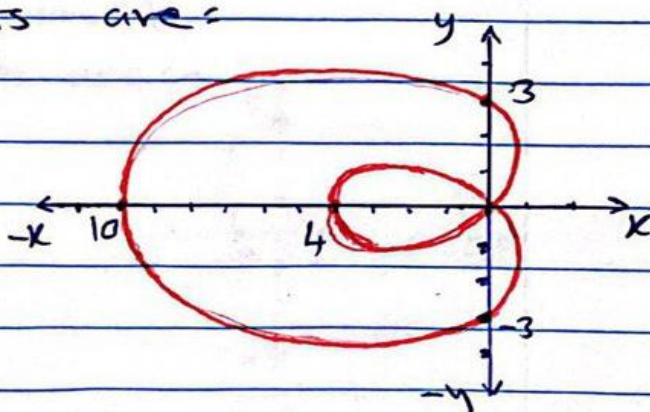
$\cos \theta \rightarrow$ located on the x-axis and symmetric on it

The Four intercepts are:

$$a = \boxed{-3}$$

$$a + b = 3 + 7 = \boxed{10}$$

$$|a - b| = |3 - 7| = \boxed{4}$$



⑧ Sketch $r = 3 + 3 \cos \theta$

Sol

compare to $r = a + b \cos \theta \rightarrow$ "limasans"

$a = 3, b = 3 \rightarrow a = b \rightarrow$ "cardioid"

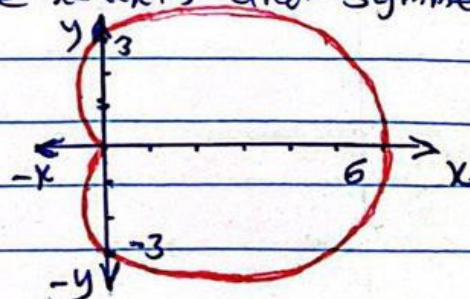
$+ \cos \theta \rightarrow$ located on the x-axis and symmetric on it.

The intercepts are:

$$a = \boxed{-3}$$

$$a + b = 6$$

$$a - b = \text{zero}$$





① Graph $r = 3 + 2 \cos \theta$
Sol:

$$a = 3, b = 2 \Rightarrow a > b \Rightarrow \text{Case III}$$

$$\frac{3}{2} = 1.5 < 2 \Rightarrow \text{"Dimple cardioid"}$$

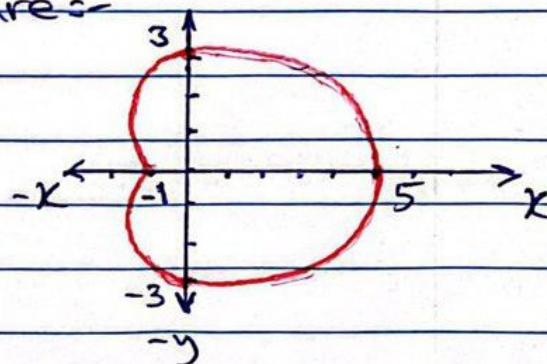
$+ \cos \theta \Rightarrow$ located on the x axis & Symmetric on it.

The Four intercepts are:-

$$a = \boxed{3}$$

$$a + b = 3 + 2 = \boxed{5}$$

$$|a - b| = |3 - 2| = \boxed{1}$$



H.W

① Graph $r = -2 + 3 \sin \theta$

② Graph $r = 5 - 2 \sin \theta$

③ sketch $r = 7 + 3 \cos \theta$

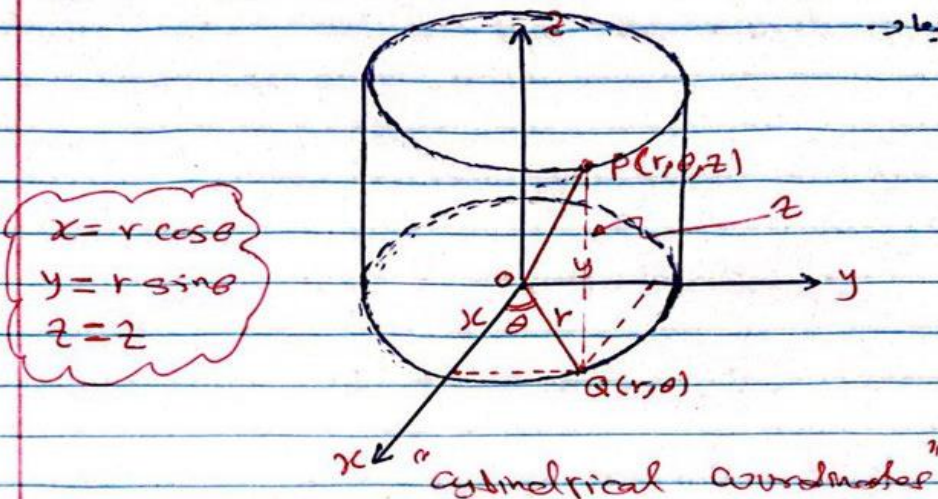
⑤ sketch $r = 4 \cos 3\theta$, $r = 4 \sin 2\theta$,
 $r^2 = 2^2 \cos 2\theta$, $r^2 = -3^2 \cos 2\theta$



Cylindrical Coordinates النقاط في الإحداثيات الأسطوانية

Cylindrical coordinates are a simple extension of two-dimensional polar coordinates to three dimensions.

الإحداثيات الأسطوانية هي امتداد بسيط للإحداثيات القطبية ثنائية الأبعاد إلى ثلاثة أبعاد.



$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta \\z &= z\end{aligned}$$

In terms of the Cartesian coordinates (x, y, z) similar to the polar coordinates:

$$r = \sqrt{x^2 + y^2} \quad , \quad \begin{array}{c} \text{Ranges} \\ r \rightarrow [0, \infty) \end{array}$$

$$\theta = \tan^{-1} \frac{y}{x} \quad , \quad \theta \rightarrow [0, 2\pi)$$

$$z = z \quad , \quad z \rightarrow (-\infty, \infty)$$

The inverse tangent " $\tan^{-1}$ " must be suitably defined to take the correct quadrant of (x, y) into account.

يجب تعريف الدالة العكسية للظل بشكل مناسب ليقبل الربع الذي تقع فيه النقطة (x, y) .



Ex 1 ① Convert the point $(1, 1, 3)$ to cylindrical coordinates (r, θ, z)

Sol.

In this example, we have given a point in (x, y, z) & asked to put it in (r, θ, z) , & to do this we need to use the equations we showed in previous page.

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1} \frac{y}{x}, \quad z = z$$

$$(1, 1, 3) \Rightarrow x=1, y=1, \boxed{z=3}$$

$$\therefore r^2 = 1^2 + 1^2 = 2 \Rightarrow \boxed{r = \sqrt{2}}$$

$$\theta = \tan^{-1} \frac{1}{1} = \tan^{-1} 1 = \boxed{45^\circ} = \boxed{\frac{\pi}{4}}$$

$\therefore (1, 1, 3)$ in Cartesian equals to $(\sqrt{2}, \frac{\pi}{4}, 3)$ in cylindrical coordinate

Ex 2 ② Convert the point $(4, \pi/4, -2)$ to Cartesian coordinates / rectangular coordinates

Sol.

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z \end{aligned}$$

$$(r, \theta, z) \rightarrow (x, y, z)$$

$$\left(\underset{r}{4}, \underset{\theta}{\pi/4}, \underset{z}{-2} \right)$$

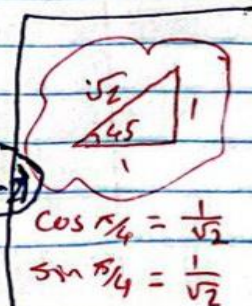
$$\begin{aligned} r &= 4 \\ \theta &= \pi/4 \\ z &= -2 \end{aligned}$$

$$\therefore x = 4 \cos \pi/4 = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$y = 4 \sin \pi/4 = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$z = -2$$

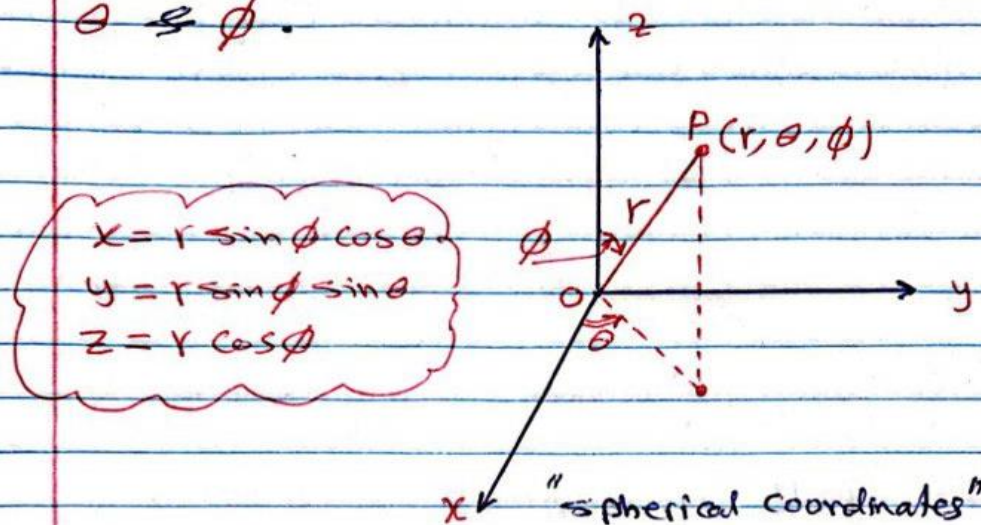
$$\Rightarrow (2\sqrt{2}, 2\sqrt{2}, -2)$$





Spherical Coordinates النقاط في الفضاء الكروي

Spherical coordinates determine the position of a point in 3-dimensional space based on the distance " r " from the origin and two angles θ & ϕ .



θ is essentially the same as the angle θ from Polar Coordinates.

$$r = \sqrt{x^2 + y^2 + z^2}$$

Range
 $r \rightarrow [0, \infty)$

$$\phi = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right)$$

$$\phi \rightarrow [0, 2\pi)$$

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

$$\theta \rightarrow [0, 2\pi)$$



Ex ③ Convert the point $(1, \sqrt{3}, 2)$ to spherical coordinates (r, θ, ϕ)

Sol.

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1} \frac{y}{x}, \quad \phi = \tan^{-1} \left(\frac{\sqrt{x^2 + y^2}}{z} \right)$$

$$(1, \sqrt{3}, 2) \rightarrow x=1, y=\sqrt{3}, z=2$$

$$r^2 = 1^2 + (\sqrt{3})^2 + (2)^2 = 8 \rightarrow r = \sqrt{8} = \boxed{2\sqrt{2}}$$

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{\sqrt{3}}{1} = \boxed{\frac{\pi}{3}}$$

$$\phi = \tan^{-1} \left(\frac{\sqrt{1^2 + (\sqrt{3})^2}}{2} \right) = \tan^{-1} \left(\frac{\sqrt{4}}{2} \right) = \tan^{-1} \frac{1}{1} = \boxed{\frac{\pi}{4}}$$

$\therefore (1, \sqrt{3}, 2)$ in Cartesian equals to $(2\sqrt{2}, \frac{\pi}{3}, \frac{\pi}{4})$ in spherical coordinates

Ex ④ Convert the point $(3, \frac{\pi}{4}, \frac{3\pi}{4})$ to Cartesian coordinates.

Sol.

$$x = r \sin \phi \cos \theta$$

$$y = r \sin \phi \sin \theta$$

$$z = r \cos \phi$$

$$(r, \theta, \phi) \rightarrow (x, y, z)$$

$$(3, \frac{\pi}{4}, \frac{3\pi}{4})$$

$$r = 3$$

$$\theta = \frac{\pi}{4}$$

$$\phi = \frac{3\pi}{4}$$

$$x = 3 \cdot \sin \frac{3\pi}{4} \cdot \cos \frac{\pi}{4} = 3 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \boxed{\frac{3}{2}}$$

$$y = 3 \cdot \sin \frac{3\pi}{4} \cdot \sin \frac{\pi}{4} = 3 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \boxed{\frac{3}{2}}$$

$$z = 3 \cos \frac{3\pi}{4} = 3 \cdot \frac{-1}{\sqrt{2}} = \boxed{-\frac{3}{\sqrt{2}}}$$

$$\left(\frac{3}{2}, \frac{3}{2}, -\frac{3}{\sqrt{2}} \right)$$



-- نهاية محاضرة " Polar Coordinates, Polar Functions, Graph of Polar Functions, Polar Coordinates and the Relations with Cartesian, Cylindrical, and Spherical Coordinated, رسم الدوال القطبية، الاحداثيات القطبية وعلاقتها مع الاحداثيات الكارتيزية والاسطوانية والكروية"--