Al-Mustaqbal University

College of Engineering and Technology

Department of Biomedical Engineering

Stage: Second

Electric circuit II

2023-2024

Lecture (8): Series A.C

9.4 IMPEDANCE AND ADMITTANCE

In the preceding section, we obtained the voltage-current relations for the three passive elements as

$$V = RI,\ V = j\omega L I,\ V = I / j\omega C \quad (9.24)$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor current as

$$V / I = R,\ V / I = j\omega L,\ V / I = 1 / j\omega C \quad (9.25)$$

From these three expressions, we obtain Ohm's law in phasor form for any type of element as

$$Z = V/I \text{ or } V = ZI \quad (9.26)$$

where Z is a frequency-dependent quantity known as impedance, measured in ohms.

The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms (Ω).

The impedance represents the opposition which the circuit exhibits to the flow of sinusoidal current. Although the impedance is the ratio of two phasors, it is not a phasor, because it does not correspond to a sinusoidally varying quantity. The impedances of resistors, inductors, and capacitors can be readily obtained from Eq. (9.25). we notice that $Z_L = j\omega L$ and $Z_C = -j/\omega C$. Consider two extreme cases of angular frequency. When $\omega = 0$ (i.e., for dc sources), $Z_L = 0$ and $Z_C \to \infty$, confirming what we already know— that the inductor acts like a short circuit, while the capacitor acts like an open circuit. When $\omega \to \infty$ (i.e., for high frequencies), $Z_L \to \infty$ and $Z_C = 0$, indicating that the inductor is an open circuit to high frequencies, while the capacitor is a short circuit.

As a complex quantity, the impedance may be expressed in rectangular form as

$$Z = R + jX \quad (9.27)$$

where R = Re Z is the resistance and X = Im Z is the reactance. The reactance X may be positive or negative. The impedance, resistance, and reactance are all measured in ohms. The impedance may also be expressed in polar form as

$$Z = |Z| \angle\theta \quad (9.28)$$

Comparing Eqs. (9.27) and (9.28), we infer that

$$Z = R + jX = |Z| \angle\theta \quad (9.29)$$

Where $\quad |Z| = \sqrt{R^2 + X^2}\,,\ \theta = \tan^{-1}\frac{X}{R}$ (9.30)

and

$$R = |Z| \cos\theta, \quad X = |Z| \sin\theta \qquad (9.31)$$

It is sometimes convenient to work with the reciprocal of impedance, known as admittance.

The admittance Y is the reciprocal of impedance, measured in siemens (S).

The admittance Y of an element (or a circuit) is the ratio of the phasor current through it to the phasor voltage across it, or

$$Y = 1/Z = I/V \qquad (9.32)$$

The admittances of resistors, inductors, and capacitors can be obtained from Eq. (9.39). As a complex quantity, we may write Y as

$$Y = G + jB \qquad (9.33)$$

where G = Re Y is called the conductance and B = Im Y is called the susceptance. Admittance, conductance, and susceptance are all expressed in the unit of siemens (or mhos). From Eqs. (9.27) and (9.33),

$$G + jB = \frac{1}{R + jX} \qquad (9.34)$$

the real and imaginary parts gives

$$G = \frac{R}{R^2 + X^2}, \quad B = \frac{-X}{R^2 + X^2} \qquad (9.35)$$

showing that $G \neq 1/R$ as it is in resistive circuits. Of course, if X = 0, then G = 1/R.

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<u>Example 9.4:</u> Find v(t) and i(t) in the circuit shown in Fig. 9.9.

<u>Solution:</u> From the voltage source 10 cos 4t , ω = 4, Vs = 10 ∠0 ◦ V The impedance is

$$Z = 5 + \frac{1}{j\omega C} = 5 + \frac{1}{j4 \times 0.1} = 5 - j2.5\ \Omega$$

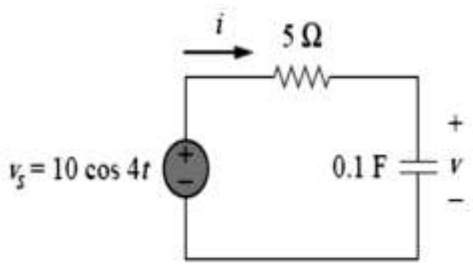


Fig. 9.9

Hence the current

$$I = \frac{Vs}{Z} = \frac{10\angle 0^\circ}{5 - j\,2.5} = \frac{10(5 + j2.5)}{5^2 + 2.5^2}$$

$$= 1.6 + j0.8 = 1.789\angle 26.57^\circ \text{ A} \qquad (9.4.1)$$

The voltage across the capacitor is

$$V = I\,ZC = \frac{I}{j\omega C} = \frac{1.789\angle 26.57^\circ}{j4 \times 0.1}$$

$$= \frac{1.789\angle 26.57^\circ}{0.4\angle 90^\circ} = 4.47\angle -63.43^\circ \text{ V} \qquad (9.4.2)$$

Converting I and V in Eqs. (9.9.1) and (9.9.2) to the time domain, we get

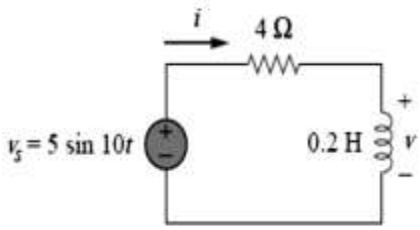
$$i(t) = 1.789 \cos(4t + 26.57^\circ)\ \text{A}$$

$$v(t) = 4.47 \cos(4t - 63.43^\circ)\ \text{V}$$

Notice that i(t) leads v(t) by 90° as expected.

Practice problem 9.4: Refer to Figure below. Determine v(t) and i(t)

Answer: 2.236 sin(10t + 63.43°) V, 1 .118 sin(10t − 26.57°) A.



9.5 KIRCHHOFF'S LAWS IN THE FREQUENCY DOMAIN

We cannot do circuit analysis in the frequency domain without Kirchhoff's current and voltage laws. Therefore, we need to express them in the frequency domain.

For KVL, let $v_1, v_2, \ldots, v_n$ be the voltages around a closed loop. Then

$$v_1 + v_2 + \cdots + v_n = 0 \qquad (9.36)$$

In the sinusoidal steady state, each voltage may be written in cosine form, so that Eq. (9.36) becomes

$$V_{m1} \cos(\omega t + \theta_1) + V_{m2} \cos(\omega t + \theta_2) + \cdots + V_{mn} \cos(\omega t + \theta_n) = 0 \qquad (9.37)$$

This can be written as

$$\text{Re}(V_{m1} e^{j\theta_1} e^{j\omega t}) + \text{Re}(V_{m2} e^{j\theta_2} e^{j\omega t}) + \cdots + \text{Re}(V_{mn} e^{j\theta_n} e^{j\omega t}) = 0$$

or

$$\text{Re}[(V_{m1} e^{j\theta_1} + V_{m2} e^{j\theta_2} + \cdots + V_{mn} e^{j\theta_n}) e^{j\omega t}] = 0 \qquad (9.37)$$

If we let $V_k = V_{mk} e^{j\theta k}$, then

$$\text{Re}[(V_1 + V_2 + \cdots + V_n) e^{j\omega t}] = 0 \qquad (9.38)$$

Since $e^{j\omega t} \neq 0$,

$$V_1 + V_2 + \cdots + V_n = 0 \qquad (9.39)$$

indicating that Kirchhoff's voltage law holds for phasors.

By following a similar procedure, we can show that Kirchhoff's current law holds for phasors. If we let $i_1, i_2, \ldots, i_n$ be the current leaving or entering a closed surface in a network at time t , then

$$\mathbf{i_1 + i_2 + \cdots + i_n = 0} \quad (9.40)$$

If $I_1, I_2, \ldots, I_n$ are the phasor forms of the sinusoids $i_1, i_2, \ldots, i_n$, then

$$\mathbf{I_1 + I_2 + \cdots + I_n = 0} \quad (9.41)$$

which is Kirchhoff's current law in the frequency domain. Once we have shown that both KVL and KCL hold in the frequency domain, it is easy to do many things, such as impedance combination, nodal and mesh analyses, superposition, and source transformation.

9.6 IMPEDANCE COMBINATIONS

Consider the N series-connected impedances shown in Fig. 9.10. The same current I flows through the impedances. Applying KVL around the loop gives

$$\mathbf{V = V_1 + V_2 + \cdots + V_N = I(Z_1 + Z_2 + \cdots + Z_N)} \quad (9.42)$$

The equivalent impedance at the input terminals is

$$\mathbf{Z_{eq} = Z_1 + Z_2 + \cdots + Z_N} \quad (9.43)$$

showing that the total or equivalent impedance of series-connected impedances is the sum of the individual impedances. This is similar to the series connection of resistances.

We can use voltage-division relationship to calculate voltage across each impedance, then

$$V_N = \frac{VZ_N}{Z_1 + Z_2 + \cdots + Z_N} \quad (9.44)$$

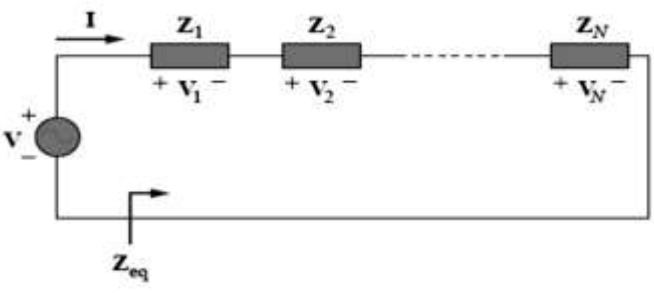


Figure 9.10 *N* impedances in series.

In the same manner, we can obtain the equivalent impedance or admittance of the N parallel-connected impedances shown in Fig. 9.19. The voltage across each impedance is the same. Applying KCL at the top node,

$$I = I_1 + I_2 + \cdots + I_N = V\left(\frac{1}{Z1} + \frac{1}{Z2} + \cdots + \frac{1}{ZN}\right) \quad (9.45)$$

The equivalent impedance is

$$\frac{1}{Zeq} = \frac{I}{V} = \frac{1}{Z1} + \frac{1}{Z2} + \cdots + \frac{1}{ZN} \quad (9.46)$$

and the equivalent admittance is

$$\mathbf{Yeq = Y1 + Y2 + \cdots + YN} \qquad (9.47)$$

This indicates that the equivalent admittance of a parallel connection of admittances is the sum of the individual admittances.

When N = 2, as shown in Fig. 9.20, the currents in the impedances are

$$I1 = \frac{Z2}{Z1 + Z2} I, \quad I2 = \frac{Z1}{Z1 + Z2} I \qquad (9.48)$$

which is the current-division principle.

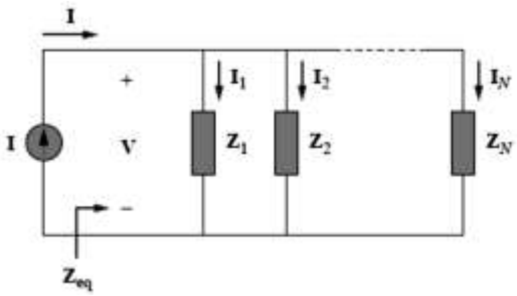


Figure 9.19 N impedances in parallel.

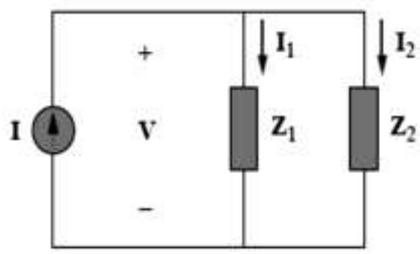


Figure 9.20 Current division.

The delta-to-wye and wye-to-delta transformations that we applied to resistive circuits are also valid for impedances. With reference to Fig. 9.21, the conversion formulas are as follows.

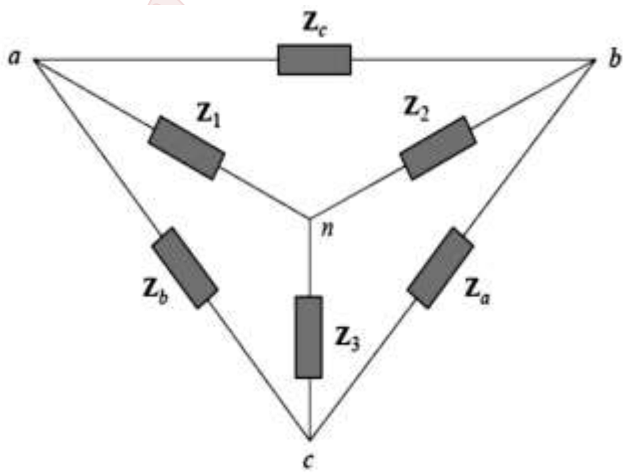


Figure 9.21 Superimposed Y and Δ networks.

Y -Δ Conversion:

$$Za = \frac{Z1Z2 + Z2Z3 + Z3Z1}{Z1}$$

$$Zb = \frac{Z1\,Z2 + Z2\,Z3 + Z3\,Z1}{Z2}$$

$$Zc = \frac{Z1\,Z2 + Z2\,Z3 + Z3\,Z1}{Z3} \qquad (9.49)$$

Δ-Y Conversion:

$$Z1 = \frac{Zb\,Zc}{Za + Zb + Zc}$$

$$Z2 = \frac{Zc\,Za}{Za + Zb + Zc}$$

$$Z3 = \frac{ZaZb}{Za + Zb + Zc} \qquad (9.50)$$

A delta or wye circuit is said to be balanced if it has equal impedances in all three branches.

When a Δ-Y circuit is balanced, Eqs. (9.49) and (9.50) become

$$Z\Delta = 3ZY \quad \text{or} \quad ZY = \frac{1}{3} Z\Delta \qquad (9.51)$$

where ZY = Z1 = Z2 = Z3 and ZΔ = Za = Zb = Zc.

Example 9.5: Find the input impedance of the circuit in Fig. 9.23. Assume that the circuit operates at ω = 50 rad/s.

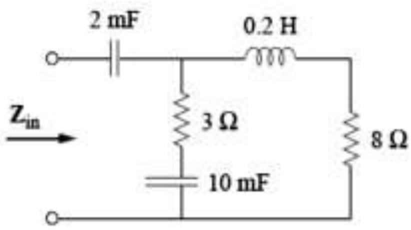


Figure 9.22

Solution:

Let

Z1 = Impedance of the 2-mF capacitor

Z2 = Impedance of the 3-Ω resistor in series with the 10-mF capacitor

Z3 = Impedance of the 0.2-H inductor in series with the 8-Ω resistor

Then

$$Z1 = \frac{1}{j\omega C} = \frac{1}{j\,50 \times 2 \times 10^{-3}} = -j10\ \Omega$$

$$Z2 = 3 + \frac{1}{j\omega C} = 3 + \frac{1}{j\,50 \times 10 \times 10^{-3}} = (3 - j2)\ \Omega$$

Z3 = 8 + jωL = 8 + j50 × 0.2 = (8 + j 10) Ω

The input impedance is

$$Zin = Z1 + Z2 \,||\, Z3 = -j\,10 + \frac{(3 - j2)(8 + j10)}{11 + j\,8}$$

$$= -j\,10 + \frac{(44 + j14)(11 - j8)}{11^2 + 8^2} = -j\,10 + 3.22 - j1.07\ \Omega$$

Thus,

Zin = 3.22 − j11.07 Ω

Practice problem 9.5: Determine the input impedance of the circuit in Fig. 9.24 at ω = 10 rad/s.

Answer: 32.38 − j73.76 Ω.

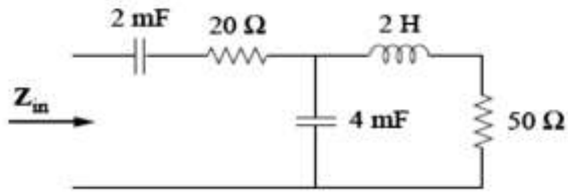


Fig. 9.24

Example 9.6: Determine vo(t) in the circuit in Fig. 9.25.

Solution: To do the analysis in the frequency domain, we must first transform the time domain circuit in Fig. 9.25 to the phasor-domain equivalent in Fig. 9.26. The transformation produces

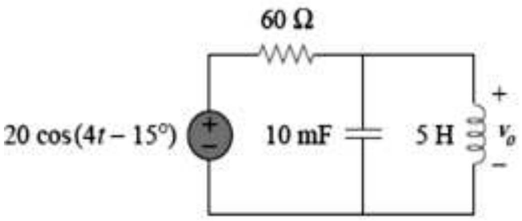


Figure 9.25 For Example 9.11.

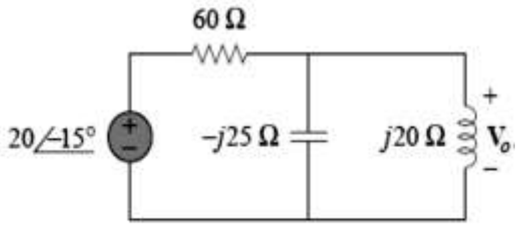


Figure 9.26 The frequency-domain equivalent of the circuit in Fig. 9.25.

$$v_s = 20\cos(4t - 15^\circ) \quad \Rightarrow \quad V_s = 20\angle{-15^\circ}\ \text{V},\ \omega = 4$$

$$10\ \text{mF} \quad \Rightarrow \quad \frac{1}{j\omega C} = \frac{1}{j4 \times 10 \times 10^{-3}} = -j\,25\ \Omega$$

$$5\text{H} \quad \Rightarrow \quad j\omega L = j\,4 \times 5 = j\,20\ \Omega$$

Let

Z1 = Impedance of the 60-Ω resistor

Z2 = Impedance of the parallel combination of the 10-mF capacitor and the 5-H inductor

Then Z 1 = 60 Ω and

$$Z2 = -j\,25 \,||\, j\,20 = \frac{-j25 \times j\,20}{-j25 + j\,20} = j\,100\ \Omega$$

By the voltage-division principle,

$$Vo = \frac{Z2}{Z1 + Z2}\ Vs = \frac{j100}{60 + j\,100}\ (20\ -15^\circ)$$

$$= (0.8575\angle 30.96^\circ)\,(20\angle{-15^\circ}) = 17.15\angle 15.96^\circ\ \text{V}.$$

We convert this to the time domain and obtain

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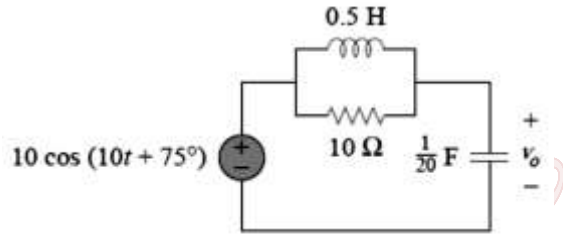
vo (t) = 17.15 cos(4t + 15.96°)V

Practice problem 9.6: Calculate vo in the circuit in Figure below.

Answer: vo (t) = 7.071 cos(10t − 60°) V.



Example 9.7: Find current I in the circuit in Fig. 9.27.

Solution:

The delta network connected to nodes a, b, and c can be converted to the Y network of Fig. 9.28. We obtain the Y impedances as follows using Eq. (9.49):

$$Zan = \frac{j4(2-j4)}{j4+2-j4+8} = \frac{4(4+j2)}{10} = (1.6 + j0.8)\ \Omega$$

$$Zbn = \frac{j4(8)}{10} = j3.2\ \Omega, \qquad Zcn = \frac{8(2-j4)}{10} = (1.6 - j\,3.2)\ \Omega$$

The total impedance at the source terminals is

$$Z = 12 + Zan + (Zbn - j3) \parallel (Zcn + j6 + 8)$$

$$= 12 + 1.6 + j0.8 + (j0.2) \parallel (9.6 + j2.8)$$

$$= 13.6 + j0.8 + \frac{j\,0.2(9.6+j2.8)}{9.6+j3}$$

$$= 13.6 + j1 = 13.64\ \angle 4.204^\circ$$

The desired current is

$$I = \frac{V}{Z} = \frac{50\ \angle 0^\circ}{13.64\ \angle 4.204^\circ} = 3.666\ \angle -4.204^\circ\ A$$

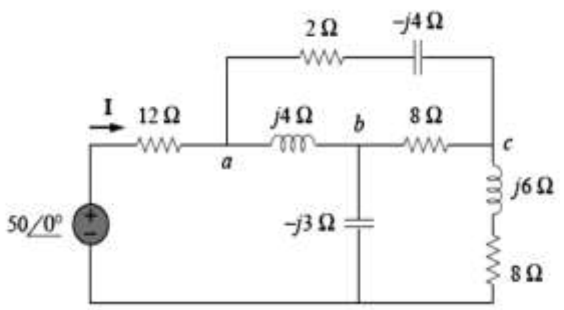


Figure 9.27

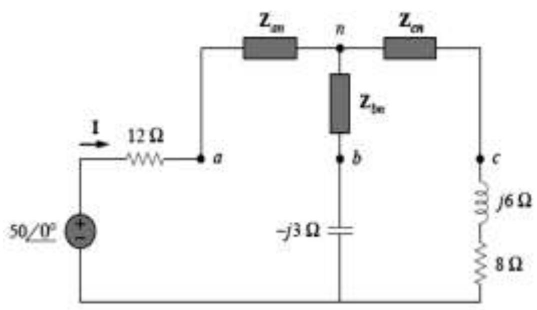
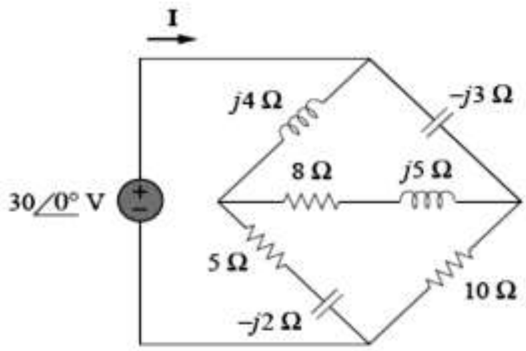


Figure 9.28 The circuit in Fig. 9.27 after delta-to-wye transformation.

Practice problem 9.8: Find I in the circuit in Figure below.

Answer: **6.364 ∠3.802° A.**



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