Discrete Mathematics

Lecture 6

Sets and set operations: cont.

Functions.

By

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Sets - review

- **A subset of B:**
  - A is a subset of B if all elements in A are also in B.
- **Proper subset:**
  - A is a proper subset of B, if A is a subset of B and $A \neq B$
- **A power set:**
  - The power set of A is a set of all subsets of A

Sets - review

- Cardinality of a set A:
  - The number of elements of in the set
- An n-tuple
  - An ordered collection of n elements
- Cartesian product of A and B
  - A set of all pairs such that the first element is in A and the second in B

Set operations

Set union:

- $A = \{1,2,3,6\}$ $B = \{2,4,6,9\}$
- $A \cup B = \{1,2,3,4,6,9\}$

Set intersection:

- $A = \{1,2,3,6\}$ $B = \{2, 4, 6, 9\}$
- $A \cap B = \{2, 6\}$

Set difference:

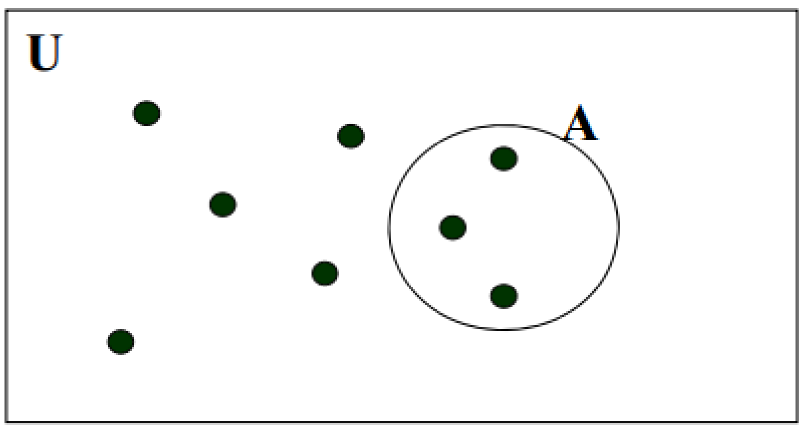
- $A = \{1,2,3,6\}$ $B = \{2, 4, 6, 9\}$
- $A - B = \{1, 3\}$
- $B – A = \{4, 9\}$

Complement of a set

Definition: Let U be the **universal set**: the set of all objects under the consideration.

Definition: The **complement of the set A**, denoted by $\overline{A}$, is the complement of A with respect to U.

- Alternate: $\overline{A} = \{ x \mid x \notin A \}$



Example: U={1,2,3,4,5,6,7,8} A ={1,3,5,7}

- $\overline{A} = \{2,4,6,8\}$

Set identities

Set Identities (analogous to logical equivalences)

- **Identity**
  - $A \cup \varnothing = A$
  - $A \cap U = A$
- **Domination**
  - $A \cup U = U$
  - $A \cap \varnothing = \varnothing$
- **Idempotent**
  - $A \cup A = A$
  - $A \cap A = A$

Set identities

- **Double complement**
  - $\overline{\overline{A}} = A$
- **Commutative**
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- **Associative**
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Distributive**
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Set identities

- **DeMorgan**
  - $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
  - $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
- **Absorbtion Laws**
  - $A \cup (A \cap B) = A$
  - $A \cap (A \cup B) = A$
- **Complement Laws**
  - $A \cup \overline{A} = U$
  - $A \cap \overline{A} = \emptyset$

Generalized unions and itersections

Definition: The **union of a collection of sets** is the set that contains those elements that are members of at least one set in the collection.

$$\bigcup_{i=1}^{n} A_i = \{A_1 \cup A_2 \cup \ldots \cup A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ i =1,2,…,n
- $$\bigcup_{i=1}^{n} A_i = \{1, 2, \ldots, n\}$$

Generalized unions and intersections

Definition: The **intersection of a collection of sets** is the set that contains those elements that are members of all sets in the collection.

$$\bigcap_{i=1}^{n} A_i = \{A_1 \cap A_2 \cap \ldots \cap A_n\}$$

Example:

- Let $A_i = \{1,2,\ldots,i\}$ i =1,2,…,n

$$\bigcap_{i=1}^{n} A_i = \{1\}$$

Computer representation of sets

- **How to represent sets in the computer?**
- **One solution: Data structures like a list**
- **A better solution:**
- Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0

Example:

All possible elements: U={1 2 3 4 5}

- Assume A={2,5}
  - Computer representation: A = 01001
- Assume B={1,5}
  - Computer representation: B = 10001

Computer representation of sets

Example:

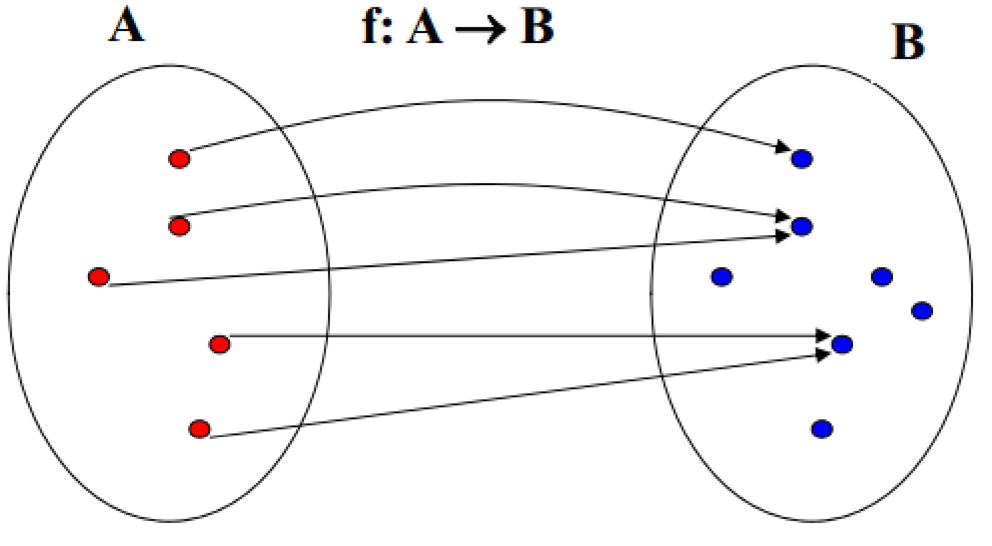
- A = 01001
- B = 10001

- The **union** is modeled with a bitwise **or**
- $A \vee B = 11001$
- The **intersection** is modeled with a bitwise **and**
- $A \wedge B = 00001$
- The **complement** is modeled with a bitwise **negation**
- $\overline{A} = 10110$

Functions

Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **$f : A \rightarrow B$** , is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



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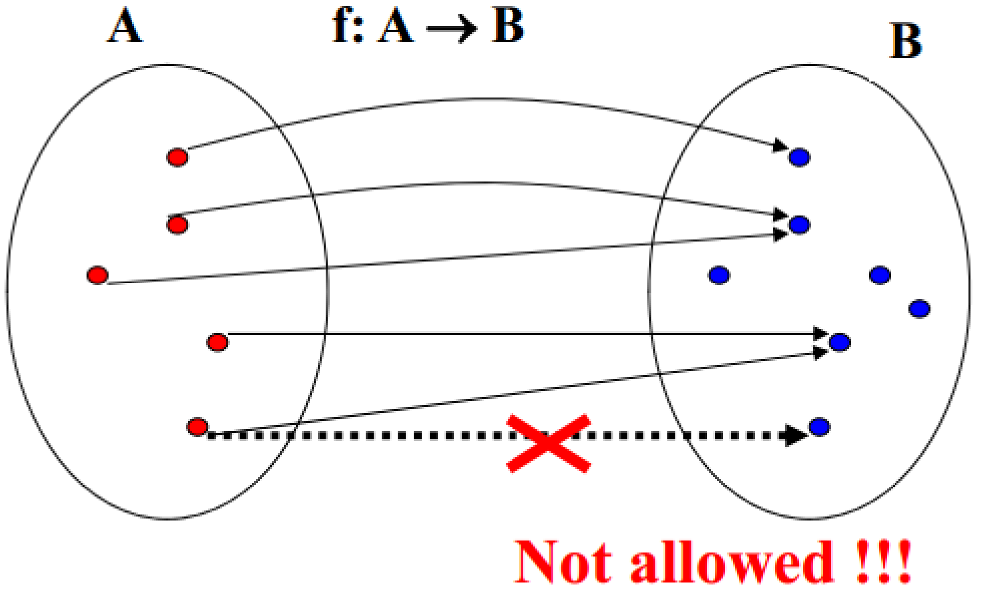


Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example1:

- Let A = {1,2,3} and B = {a,b,c}
- Assume f is defined as:
  - $1 \to c$
  - $2 \to a$
  - $3 \to c$
- Is f a function ?
- **Yes.** since f(1)=c, f(2)=a, f(3)=c. each element of A is assigned an element from B

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using ‘standard’ functions)

Example 2:

- Let A = {1,2,3} and B = {a,b,c}
- Assume g is defined as
  - $1 \rightarrow c$
  - $1 \rightarrow b$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is g a function?
- **No.** g(1) = is assigned both c and b.

Representing functions

Representations of functions:

1. Explicitly state the assignments in between elements of the two sets
2. Compactly by a formula. (using 'standard' functions)

Example 3:

- A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}
- Define h: A $\rightarrow$ B as:
  - h(x) = x mod 3.
  - (the result is the remainder after the division by 3)
- Assignments:
- 0 → 0
- 1 → 1
- 2 → 2

3 → 0

4 → 1

...

Important sets

Definitions: Let f be a function from A to B.

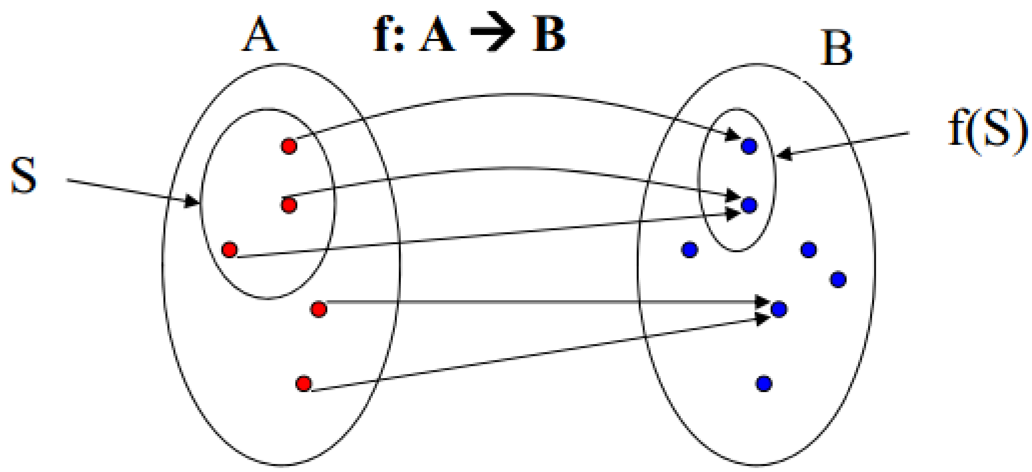
- We say that A is the **domain** of f and B is the **codomain** of f.
- If f(a) = b, **b is the image of a** and **a is a pre-image of b**.
- **The range of f** is the set of all images of elements of A. Also, if f is a function from A to B, we say f maps A to B.

Example: Let A = {1,2,3} and B = {a,b,c}

- Assume f is defined as: 1 → c, 2 → a, 3 → c
- What is the image of 1?
- 1 → c c is the image of 1
- What is the pre-image of a?
- 2 → a 2 is a pre-image of a.
- Domain of f ? {1,2,3}
- Codomain of f ? {a,b,c}
- Range of f ? {a,c}

Image of a subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.

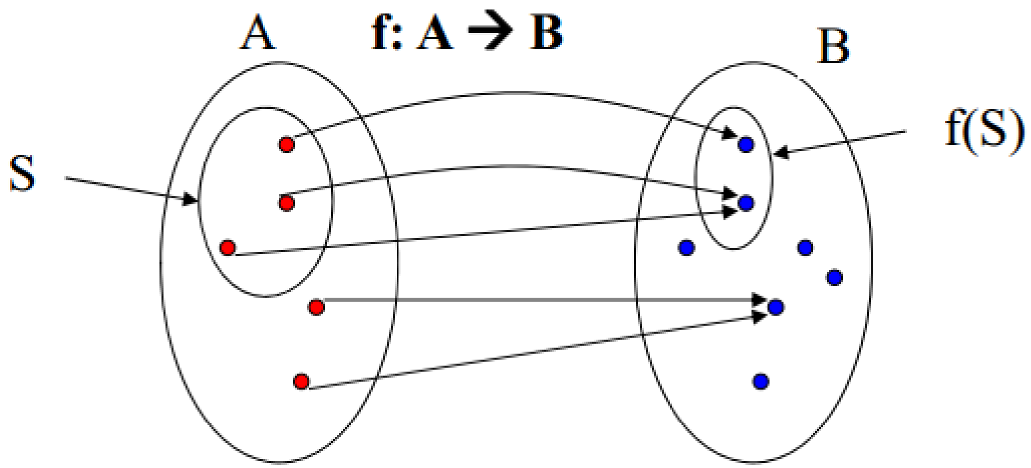


Example:

- Let A = {1,2,3} and B = {a,b,c} and f: 1 → c, 2 → a, 3 → c
- Let S = {1,3} then image f(S) = ?

Image of a subset

Definition: Let f be a function from set A to set B and let S be a subset of A. The image of S is a subset of B that consists of the images of the elements of S. We denote the image of S by f(S), so that $f(S) = \{ f(s) \mid s \in S \}$.



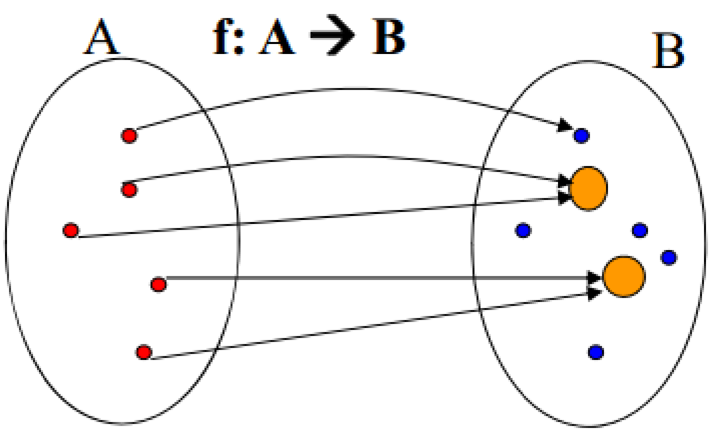
Example:

- Let A = {1,2,3} and B = {a,b,c} and f: 1 → c, 2 → a, 3 → c
- Let S = {1,3} then image f(S) = {c}.

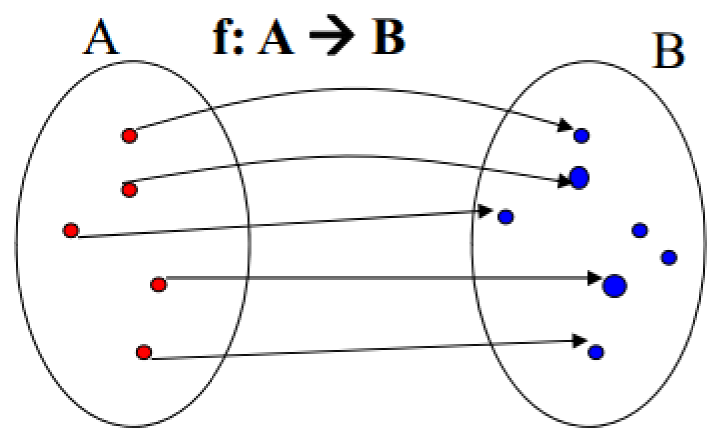
Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

Alternate: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.



Not injective



Injective function

Injective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$
- Is f one to one? **No**, it is not one-to-one since f(1) = f(3) = c, and $1 \neq 3$.

Example 2: Let $g : Z \rightarrow Z$, where g(x) = 2x - 1.

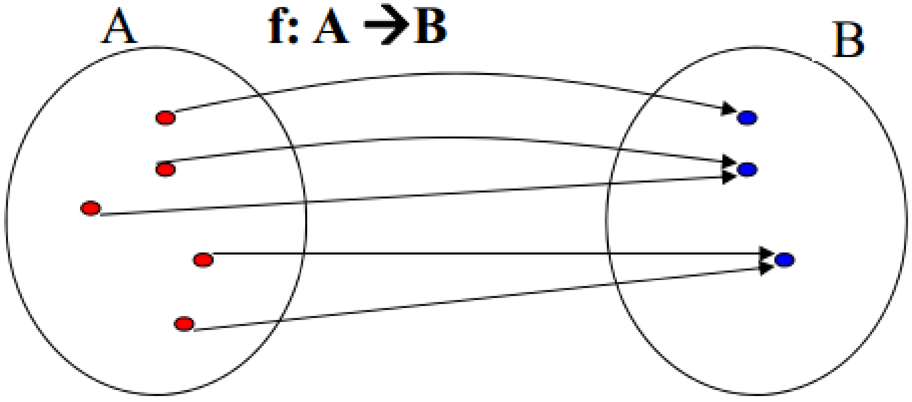
- Is g is one-to-one (why?)
- **Yes.**
- Suppose g(a) = g(b), i.e., 2a - 1 = 2b - 1 => 2a = 2b

``   => a = b.

Surjective function

Definition: A function f from A to B is called **onto**, or **surjective**, if and only if for every $b \in B$ there is an element $a \in A$ such that $f(a) = b$.

Alternative: all co-domain elements are covered



Surjective functions

Example 1: Let A = {1,2,3} and B = {a,b,c}

- Define f as
  - $1 \rightarrow c$
  - $2 \rightarrow a$
  - $3 \rightarrow c$

- Is f an onto?
- **No.** f is not onto, since $b \in B$ has no pre-image.

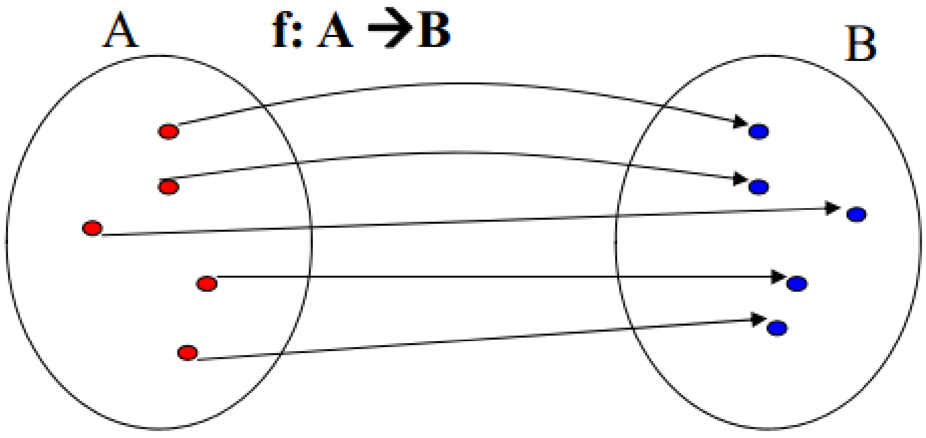
Example 2: A = {0,1,2,3,4,5,6,7,8,9}, B = {0,1,2}

- Define $h: A \rightarrow B$ as $h(x) = x \bmod 3$.

- Is h an onto function?
- **Yes.** h is onto since a pre-image of 0 is 6, a pre-image of 1 is 4, a pre-image of 2 is 8.

Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one and onto**.



Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f is a bijection? **Yes.** It is both one-to-one and onto.

- **Note:** Let f be a function from a set A to itself, where A is finite. f is one-to-one if and only if f is onto.
- This is not true for A an infinite set. Define $f : Z \rightarrow Z$, where $f(z) = 2 * z$. f is one-to-one but not onto (3 has no pre-image).

Bijective functions

Example 2:

- Define g : W → W (whole numbers), where $g(n) = \lfloor n/2 \rfloor$ (floor function).
  - $0 \to \lfloor 0/2 \rfloor = \lfloor 0 \rfloor = 0$
  - $1 \to \lfloor 1/2 \rfloor = \lfloor 1/2 \rfloor = 0$
  - $2 \to \lfloor 2/2 \rfloor = \lfloor 1 \rfloor = 1$
  - $3 \to \lfloor 3/2 \rfloor = \lfloor 3/2 \rfloor = 1$
- ...
- Is g a bijection?
  - **No.** g is onto but not 1-1 ($g(0) = g(1) = 0$ however $0 \neq 1$.

Any questions ??