Ministry of Higher Education and Scientific Research
Almustaqbal University, College of Engineering
And Engineering Technologies
Department of Electrical Engineering

Two Week :

Unit Step Forcing Function

Course Name : Electrical Circuits Analysis
Stage : Second
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Singularity Functions

A basic understanding of singularity functions will help us make sense of the response of first-order circuits to a sudden application of an independent dc voltage or current source. Singularity functions (also called ***switching functions***) are very useful in circuit analysis. They serve as good approximations to the switching signals that arise in circuits with switching operations. They are helpful in the neat, compact description of some circuit phenomena, especially the step response of RC or RL circuits .

Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

The three most widely used singularity functions in circuit analysis are :

1. the ***unit step*** function.
2. the ***unit impulse*** function.
3. the ***unit ramp*** function.

1) Unit step function

The unit step function ***u(t)*** is 0 for negative values of t and 1 for positive values oft.

The unit step function is undefined at $t = 0$, where it changes abruptly from 0 to 1 (shown in Fig.4.1). It is dimensionless, like other mathematical functions such as sine and cosine. In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \qquad (4.1)$$

If the abrupt change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, which is the same as saying that $u(t)$ is delayed by t_0 seconds, as shown in Fig.4.2(a), the unit step function becomes

$$u(t - t_o) = \begin{cases} 0, & t < t_o \\ 1, & t > t_o \end{cases} \tag{4.2}$$

If the change is at $t = -t_0$, meaning that $u(t)$ is advanced by t_0 seconds, Fig.4.2(b), the unit step function becomes

$$u(t + t_o) = \begin{cases} 0, & t < -t_o \\ 1, & t > -t_o \end{cases} \tag{4.3}$$

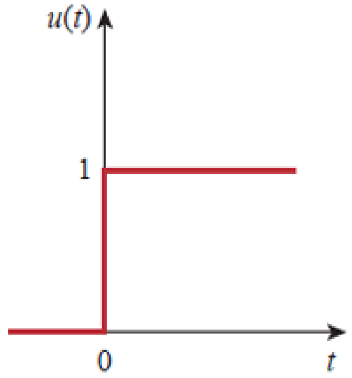


Fig.4.1 The unit step function.

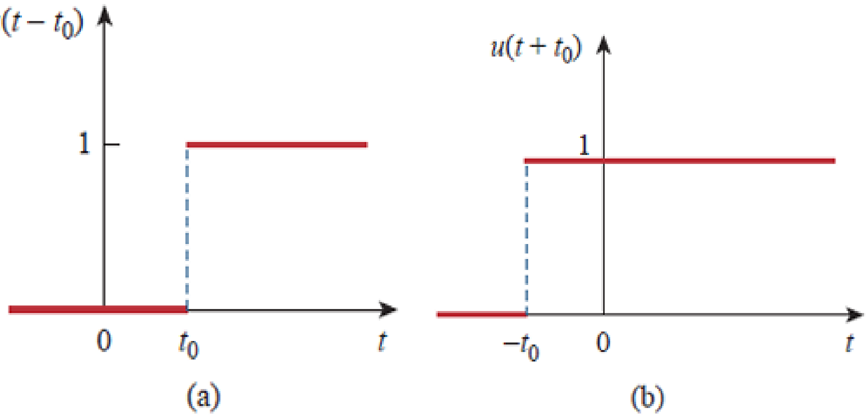


Fig.4.2 The unit step function
(a) delayed by t_0 (b) advanced by t_0.

We use the step function to represent an abrupt change in voltage or current, like the changes that occur in the circuits of control systems and digital computers. For example, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases} \tag{4.4}$$

may be expressed in terms of the unit step function as

$$v(t) = V_0 u(t - t_0) \tag{4.5}$$

If we let $t_0 = 0$ then $v(t)$is simply the step voltage $V_0 u(t)$. A voltage source of $V_0 u(t)$ is shown in Fig.4.3 (a); its equivalent circuit is shown in Fig.4.3 (b). It is evident in Fig.4.3 (a) that terminals *a*-*b* are short circuited ($v = 0$) for $t < 0$ and that $v = V_0$ appears at the terminals

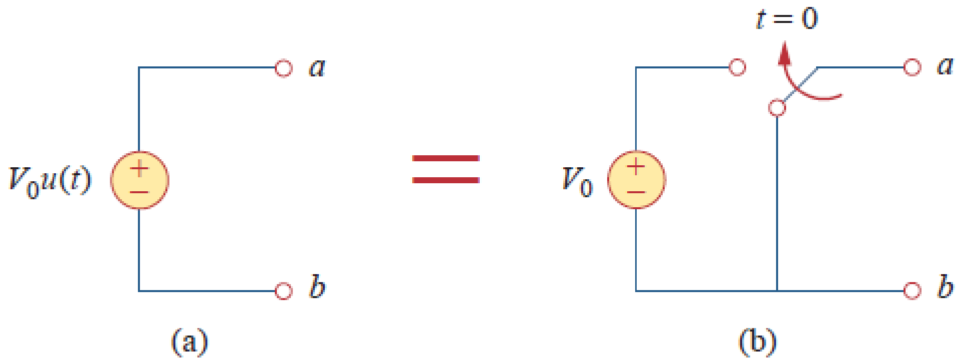


Fig.4.3 (a) A voltage source of $V_0u(t)$ (b) its equivalent circuit.

for $t > 0$. Similarly, a current source of $I_0u(t)$ is shown in Fig.4.4 (a), while its equivalent circuit is in Fig.4.4 (b). Notice that for $t < 0$, there is an open circuit ($i = 0$) , and that $i = I_0$ flows for $t > 0$.

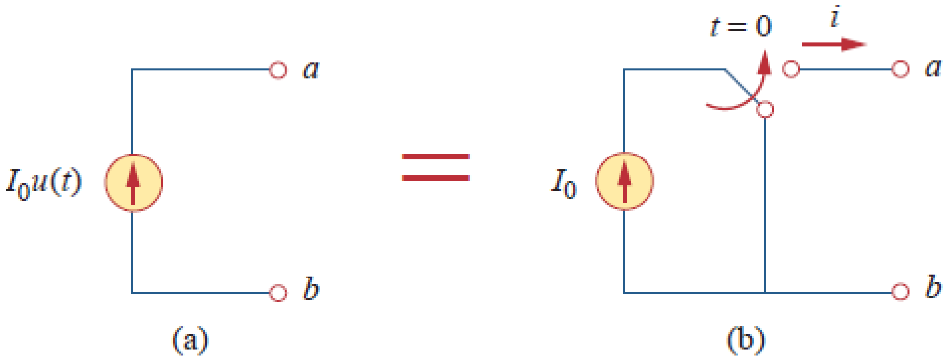


Fig.4.4 (a) A current source of $I_0u(t)$ (b) its equivalent circuit.

2) Unit impulse function

The derivative of the unit step function $u(t)$ is the ***unit impulse function*** (***or delta function***) $\boldsymbol{\delta(t)}$, which we write as

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0 & t < 0 \\ Undefined & t = 0 \\ 0 & t > 0 \end{cases} \qquad (4.6)$$

Impulsive currents and voltages occur in electric circuits as a result of switching operations or impulsive sources. Although the unit impulse function is not physically realizable (just like ideal sources, ideal resistors, etc.), it is a very useful mathematicaltool.

The unit impulse may be regarded as an applied or resulting shock. It may be visualized asa very short duration pulse of unit area. This may be expressed mathematically as

$$\int_{0^-}^{0^+} \delta(t)dt = 1 \tag{4.7}$$

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ is the time just after $t = 0$. For this reason, it is customary to write 1 (denoting unit area) beside the arrow that is used to symbolize the unit impulse function, as in Fig.4.5. The unit area is known as the *strength* of the impulse function. When an impulse function has a strength other than unity, the area of the impulse is equal to its strength. For example, an impulse function $10\delta(t)$ has an area of 10. Fig.4.6 shows the impulse functions $5\delta(t+2)$, $10\delta(t)$, and $-4\delta(t-3)$.

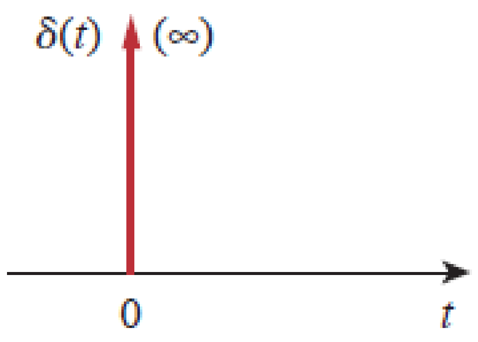


Fig.4.5 The unit impulse function.

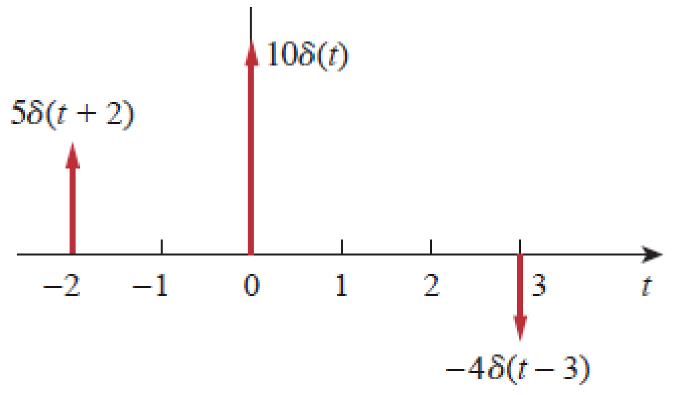


Fig.4.6 Three impulse functions.

To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_a^b f(t)\delta(t-t_0)dt \tag{4.8}$$

where $a < t_0 < b$. Since $\delta(t-t_0) = 0$ except at $t = t_0$, the integrand is zero except at t_0. Thus,

$$\int_a^b f(t)\delta(t-t_0)dt = \int_a^b f(t_0)\delta(t-t_0)dt = f(t_0)\int_a^b \delta(t-t_0)dt = f(t_0)$$

$$\int_a^b f(t)\delta(t-t_0)dt = f(t_0) \tag{4.9}$$

This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point where the impulse occurs. This is a highly useful property of the impulse function known as the **sampling** or **sifting** property. The special case of Eq. (4.8) is for $t_0 = 0$. Then Eq. (4.9) becomes

$$\int_{0^-}^{0^+} f(t)\delta(t)dt = f(0) \qquad (4.10)$$

3) Unit ramp function

Integrating the unit step function $u(t)$ results in the ***unit ramp function*** $\boldsymbol{r(t)}$;

$$r(t) = \int_{-\infty}^{t} u(t)dt = tu(t) \qquad (4.11)$$

or

$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases} \qquad (4.12)$$

The unit ramp function is zero for negative values of t and has a unit slope for positive values of t.

Fig.4.7 shows the unit ramp function. In general, a ramp is a function that changes at aconstant rate.

The unit ramp function may be delayed or advanced as shown in Fig.4.8. For the delayedunit ramp function,

$$r(t - t_0) = \begin{cases} 0, & t \leq t_0 \\ t - t_0, & r \geq t_0 \end{cases} \qquad (4.13)$$

and for the advanced unit ramp function,

$$r(t + t_0) = \begin{cases} 0, & t \leq -t_0 \\ t + t_0, & t \geq -t_0 \end{cases} \qquad (4.14)$$

We should keep in mind that the three singularity functions (impulse, step, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt} \qquad (4.15)$$

or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(t)dt, \qquad r(t) = \int_{-\infty}^{t} u(t)dt \qquad (4.16)$$

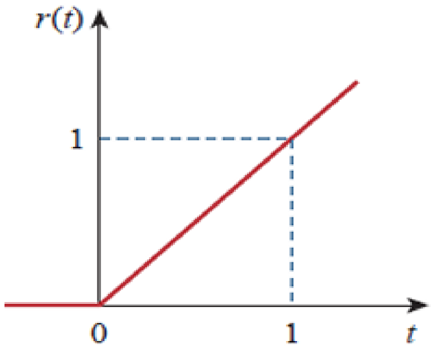


Fig.4.7 The unit ramp function.

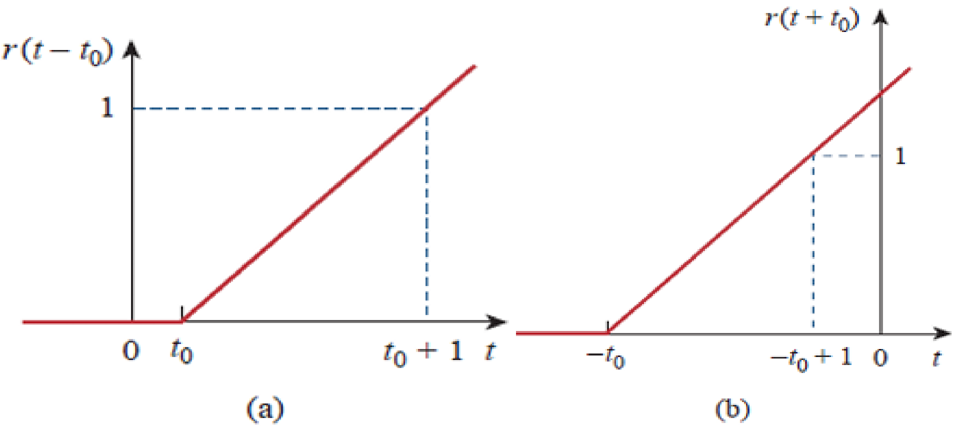


Fig.4.8 The unit ramp function
(a) delayed by t_o (b) advanced by t_o.

Example 6: Express the voltage pulse in Fig.1 in terms of the unit step. Calculate itsderivative and sketch it.

Solution: The type of pulse in Fig.1 is called the **gate function**. It may be regarded as a step function that switcheson at one value of t and switches off at another value of t. This gate function switches on at $t = 2s$ and switches off at $t = 5s$. It consists of the sum of two unit step functions as shown in Fig.2 (a). From the figure, it is evident that

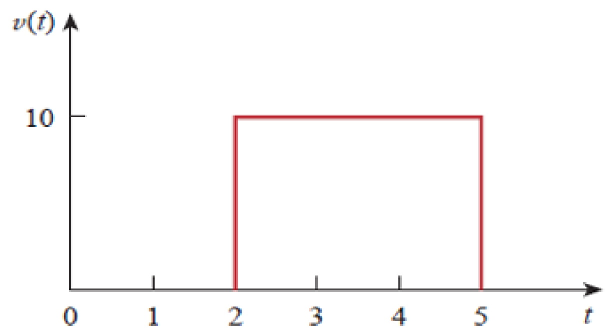


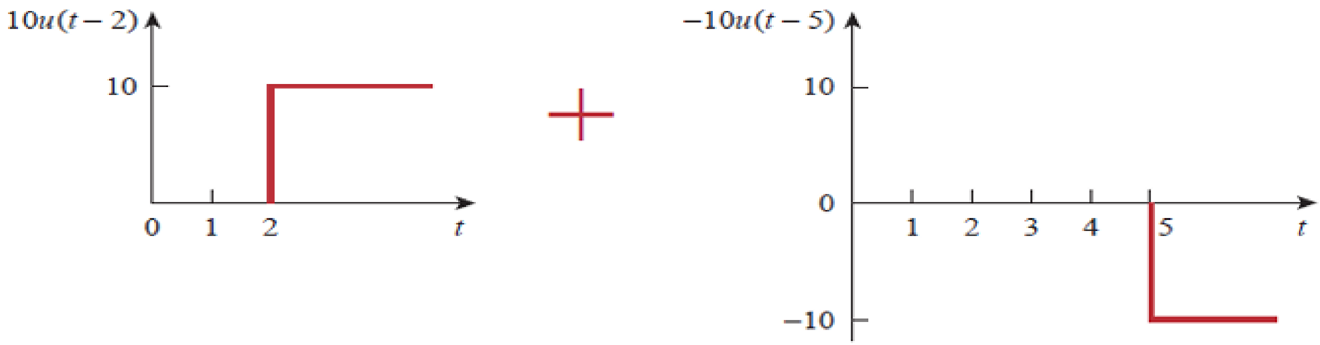
Fig.1

$$v(t) = 10u(t-2) - 10u(t-5) = 10[u(t-2) - u(t-5)]$$

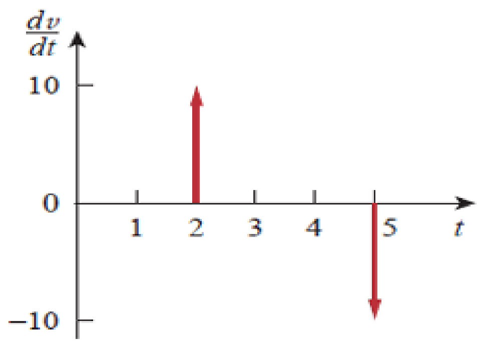
Taking the derivative of this gives

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)]$$

which is shown in Fig.2 (b). We can obtain Fig.2 (b). directly from Fig.1. by simply observing that there is a sudden increase by 10 V at $t = 2s$ leading to $10\delta(t-2)$. At $t = 5s$, there is a sudden decrease by 10 V leading to $-10V\ \delta(t-5)$.



(a)



(b)

Fig.2 (a) Decomposition of the pulse in Fig.1, (b) derivative of the pulse in Fig.1.

H.W. 6: Express the current pulse in Fig.1in terms of the unit step. Find its integral and sketch it.

Answer: $10[u(t) - 2u(t-2) + u(t-4)]$, $10[r(t) - 2r(t-2) + r(t-4)]$. See Fig.2.

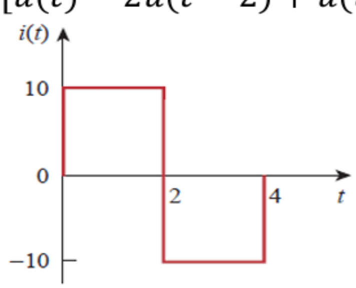


Fig.1

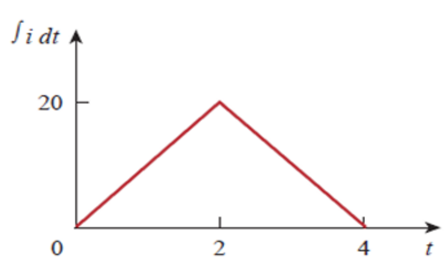


Fig.2

Example 7: Express the ***sawtooth*** function shown in Fig.1in terms of singularity functions.

Solution:

There are three ways of solving this problem. The first method is by mere observation of the given function, while the other methods involve some graphical manipulations of the function.

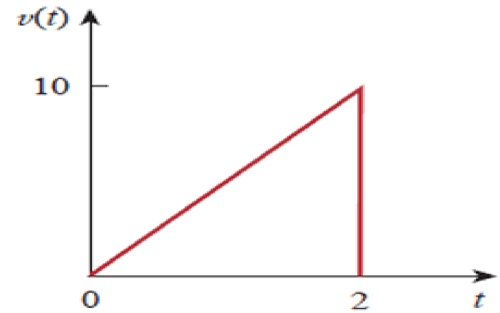


Fig. 1

METHOD 1 By looking at the sketch of $v(t)$ in Fig.1, it is not hard to notice that the given function $v(t)$ is a combination of singularity functions. So we let

$$v(t) = v_1(t) + v_2(t) + \cdots \quad (1)$$

The function $v_1(t)$ is the ramp function of slope 5, shown in Fig.2 (a); that is,

$$v_1(t) = 5r(t) \quad (2)$$

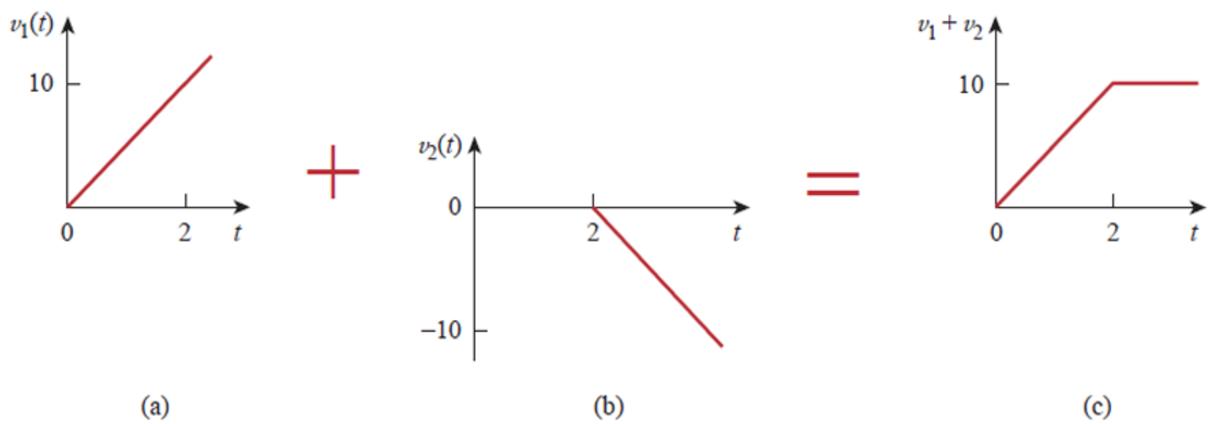


Fig.2 Partial decomposition of $v(t)$ in Fig.1.

Since $v(t)$ goes to infinity, we need another function at $t = 2s$ in order to get $v(t)$. We let this function be v_2, which is a ramp function of slope −5, as shown in Fig.2 (b); that is,

$$v_2(t) = -5r(t-2) \quad (3)$$

Adding v_1 and v_2 gives us the signal in Fig.2 (c). Obviously, this is not the same as $v(t)$ in Fig.1. But the difference is simply a constant 10 units for $t > 2s$. By adding a third signal v_3, where

$$v_3 = -10u(t-2) \quad (4)$$

we get $v(t)$, as shown in Fig.3. Substituting Eqs. (2) through (4) into Eq. (1) gives

$$v(t) = 5r(t) - 5r(t-2) - 10u(t-2)$$

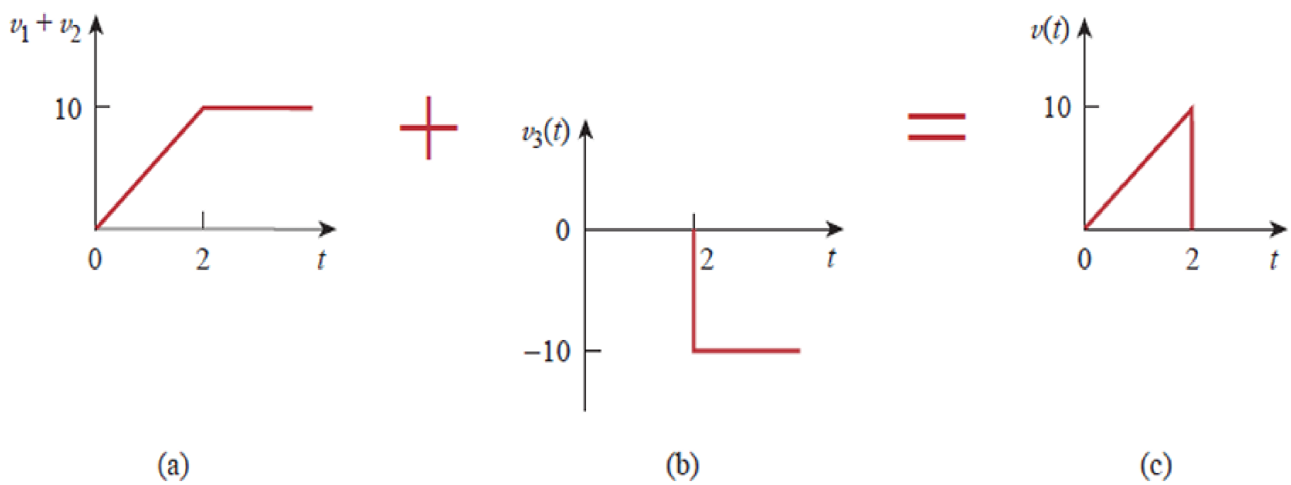


Fig.3 Complete decomposition of $v(t)$ in Fig.1.

METHOD 2 A close observation of Fig.1 reveals that $v(t)$ is a multiplication of twofunctions: a ramp function and a gate function. Thus,

$$v(t) = 5t[u(t) - u(t-2)] = 5tu(t) - 5tu(t-2)$$

$$= 5r(t) - 5(t-2+2)u(t-2) = 5r(t) - 5(t-2)u(t-2) - 10u(t-2)$$

$$= 5r(t) - 5r(t-2) - 10u(t-2)$$

the same as before.

METHOD 3 This method is similar to **Method 2**. We observe from Fig.1 that $v(t)$ is amultiplication of a ramp function and a unit step function, as shown in Fig.4. Thus,

$$v(t) = 5r(t)u(-t+2)$$

If we replace $u(-t)$ by $1 - u(t)$,then we can replace $u(-t+2)$ by $1 - u(t-2)$ .Hence,

$$v(t) = 5r(t)[1 - u(t-2)]$$

which can be simplified as in **Method 2** to get the same result.

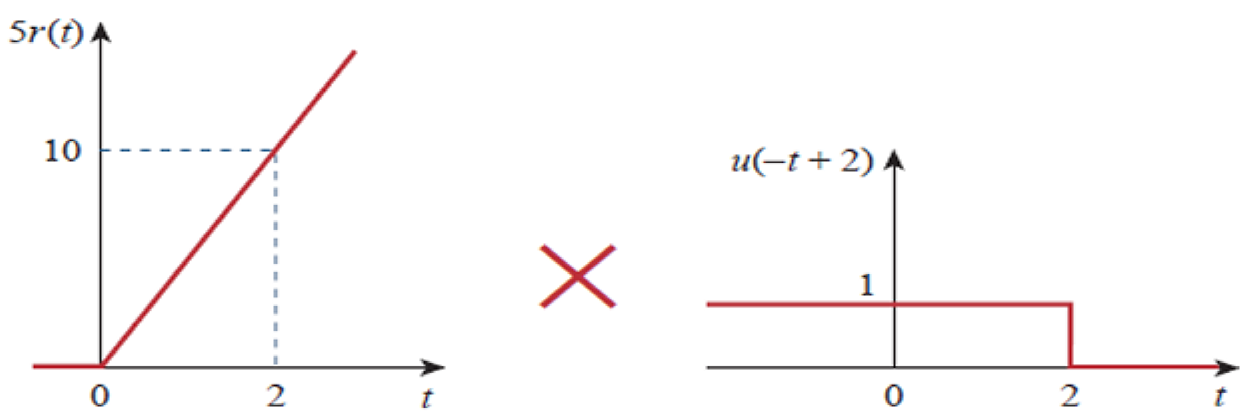


Fig.4. Decomposition of $v(t)$ in Fig.1.

H.W. 7: Refer to Fig.1. Express $i(t)$ in terms of singularity functions.

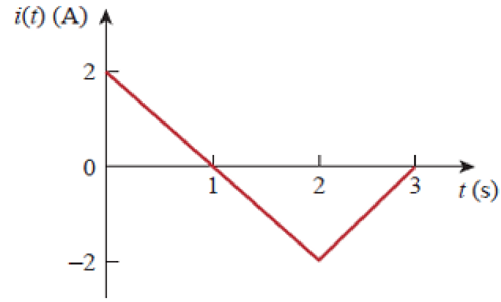


Fig.1

Answer: $2u(t) - 2r(t) + 4r(t-2) - 2r(t-3)$.

Example 8: Given the signal

$$g(t) = \begin{cases} 3, & t < 0 \\ -2, & 0 < t < 1 \\ 2t - 4, & t > 1 \end{cases}$$

express $g(t)$ in terms of step and ramp functions.

Solution:

The signal $g(t)$ may be regarded as the sum of three functions specified within the three intervals $t < 0, 0 < t < 1$, and $t > 1$.

For $t < 0$, $g(t)$ may be regarded as 3 multiplied by $u(-t)$, where $u(-t) = 1$ for $t < 0$ and 0 for $t > 0$. Within the time interval $0 < t < 1$, the function may be considered as -2 multiplied by a gated function $[u(t) - u(t-1)]$. For $t > 1$, the function may be regarded as $2t - 4$ multiplied by the unit step function $u(t-1)$. Thus,

$$g(t) = 3u(-t) - 2[u(t) - u(t-1)] + (2t-4)u(t-1)$$

$$= 3u(-t) - 2u(t) + (2t - 4 + 2)u(t-1)$$

$$= 3u(-t) - 2u(t) + 2(t-1)u(t-1) = 3u(-t) - 2u(t) + 2r(t-1)$$

One may avoid the trouble of using $u(-t)$ by replacing it with $1 - u(t)$. Then

$$g(t) = 3[1 - u(t)] - 2u(t) + 2r(t-1) = 3 - 5u(t) + 2r(t-1)$$

Alternatively, we may plot $g(t)$ and apply **Method 1** from **Example 7**.

H.W. 8: If

$$h(t) = \begin{cases} 0, & t < 0 \\ 8, & 0 < t < 2 \\ 2t+6, & 2 < t < 6 \\ 0, & t > 6 \end{cases}$$

express $h(t)$ in terms of the singularity functions.

Answer: $\mathbf{8u(t) + 2u(t-2) + 2r(t-2) - 18u(t-6) - 2r(t-6)}$

Example 9: Evaluate the following integrals involving the impulse function:

$$\int_0^{10} (t^2 + 4t - 2)\delta(t-2)dt$$

$$\int_{-\infty}^{\infty} [\delta(t-1)e^{-t}\cos t + \delta(t+1)e^{-t}\sin t]dt$$

Solution:

For the first integral, we apply the sifting property in Eq. (4.9).

$$\int_0^{10} (t^2 + 4t - 2)\delta(t-2)dt = (t^2 + 4t - 2)|_{t=2} = 4 + 8 - 2 = 10$$

Similarly, for the second integral,

$$\int_{-\infty}^{\infty} [\delta(t-1)e^{-t}\cos t + \delta(t+1)e^{-t}\sin t]dt$$

$$= e^{-t}\cos t|_{t=1} + e^{-t}\sin t|_{t=-1} = e^{-1}\cos 1 + e^{1}\sin(-1) = 0.1988 - 2.2873 = -2.0885$$

H.W. 9: Evaluate the following integrals:

$$\int_{-\infty}^{\infty} (t^3 + 5t^2 + 10)\delta(t+3)dt, \qquad \int_0^{10} \delta(t-\pi)\cos 3t dt$$

Answer: 28, −1.

Thank you very much

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