اسم المادة : رياضيات-1
اسم التدريسي : د حسين كاظم حلواص و م.م زين العابدين كريم
المرحلة : الأولى
السنة الدراسية : 2023-2024

عنوان المحاضرة:

Trigonometric Functions & Relations, Tri Equations, Graphing of Functions الدوال المثلثية وعلاقاتها، المعادلات المثلثية، رسم الدوال المثلثية

Trigonometric functions :- الدوال المثلثية

There are six trigonometric functions; they are;

1- $\sin\theta = \frac{y}{1} = y$

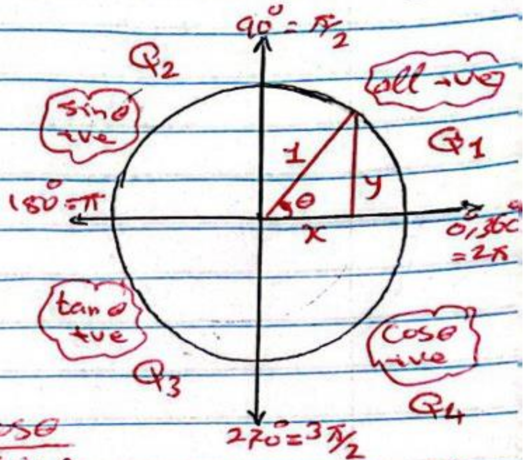
2- $\cos\theta = \frac{x}{1} = x$

3- $\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$

4- $\csc\theta = \frac{1}{y} = \frac{1}{\sin\theta}$

5- $\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta}$

6- $\cot\theta = \frac{x}{y} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$



"unit circle"

We have two angle representatives;

1- Degree.
2- Radian.

Degree	Radian	Degree	Radian
30°	π/6	300°	5π/3
45°	π/4	315°	7π/4
60°	π/3	330°	11π/6
90°	π/2	360°	2π
120°	2π/3	0°	0π = 0
135°	3π/4		
150°	5π/6		
180°	π		
210°	7π/6		
225°	5π/4		
240°	4π/3		
270°	3π/2		

The unit circle has four quarters.
The sine & cosine func in these quarters are:

① Q_1

Angle	Sine	Cosine
0	0	1
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
60°	$\sqrt{3}/2$	$1/2$
90°	1	0

② Q_2

Angle	Sine	Cosine
180°	0	-1
150°	$\frac{1}{2}$	$-\sqrt{3}/2$
135°	$\frac{1}{\sqrt{2}}$	$-1/\sqrt{2}$
120°	$\sqrt{3}/2$	$-\frac{1}{2}$

③ Q_3

Angle	Sine	Cosine
210°	$-\frac{1}{2}$	$-\sqrt{3}/2$
225°	$-1/\sqrt{2}$	$-1/\sqrt{2}$
240°	$-\sqrt{3}/2$	$-\frac{1}{2}$
220°	-1	0

④ Q_4

Angle	Sine	Cosine
300°	$-\sqrt{3}/2$	$\frac{1}{2}$
315°	$-1/\sqrt{2}$	$1/\sqrt{2}$
330°	$-\frac{1}{2}$	$\sqrt{3}/2$

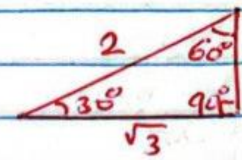
How To Evaluate The Trigonometric Funs.

① $30^\circ \Longrightarrow Q_1$

$\sin 30 = \frac{1}{2}$

$\cos 30 = \frac{\sqrt{3}}{2}$

$\tan 30 = \frac{1}{\sqrt{3}}$

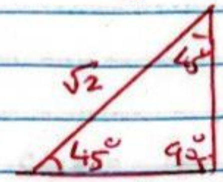


② $45^\circ \Longrightarrow Q_1$

$\sin 45 = \frac{1}{\sqrt{2}}$

$\cos 45 = \frac{1}{\sqrt{2}}$

$\tan 45 = 1$

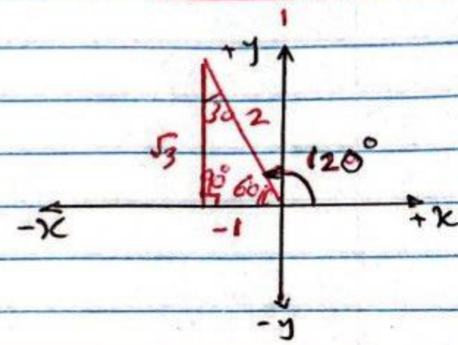


③ $120^\circ \longrightarrow Q_2$

$\sin 120 = \frac{\sqrt{3}}{2}$

$\cos 120 = -1/2$

$\tan 120 = \sqrt{3}/_{-1} = -\sqrt{3}$

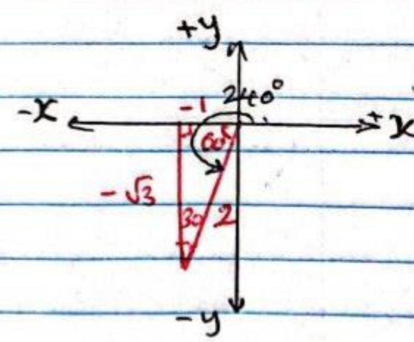


④ $240^\circ \longrightarrow Q_3$

$\sin 240 = -\sqrt{3}/2$

$\cos 240 = -1/2$

$\tan 240 = -\sqrt{3}/_{-1} = \sqrt{3}$

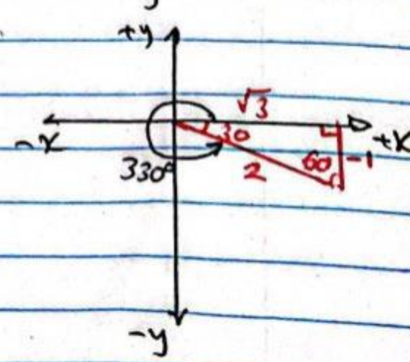


⑤ $330^\circ \longrightarrow Q_4$

$\sin 330 = -1/2$

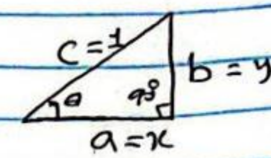
$\cos 330 = \sqrt{3}/2$

$\tan 330 = -1/\sqrt{3}$



Trigonometric Relations :- العلاقات المثلثية

* Pythagorean Theorm نظرية فيثاغورس



$$a^2+b^2=c^2$$

If $a=x$, $b=y$, $c=1$

$\therefore x^2+y^2=1$

If we know from the unit circle that

$\sin\theta = y$

$\cos\theta = x$

$$\therefore \sin^2\theta+\cos^2\theta=1 \quad \text{—— (1)}$$

- if we divided eq:1 by $\sin^2\theta$; yields,

$$1+\frac{\cos^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta} \Longrightarrow 1+\cot^2\theta=\csc^2\theta \quad \text{—— (2)}$$

- If we divided eq: 1 by $\cos^2\theta$; yields,

$$\frac{\sin^2\theta}{\cos^2\theta}+1=\frac{1}{\cos^2\theta} \Longrightarrow 1+\tan^2\theta=\sec^2\theta \quad \text{—— (3)}$$

Even & Odd Trigonometric Functions :

الدوال المثلثية الزوجية والفردية

Even Funs	Odd Funs
$\cos(-\theta)=\cos\theta$	$\sin(-\theta)=-\sin\theta$
$\sec(-\theta)=\sec\theta$	$\csc(-\theta)=-\csc\theta$
	$\tan(-\theta)=-\tan\theta$
	$\cot(-\theta)=-\cot\theta$

Trigonometric Functions are equal at:

الدوال المثلثية تتساوى عند الزوايا التالية :-

$\cos\theta = \sin(90-\theta)$
$\sin\theta = \cos(90-\theta)$
$\csc\theta = \sec(90-\theta)$
$\cot\theta = \tan(90-\theta)$

Ex)

$\cos 0 = \sin 90 = 1$
$\cos 10 = \sin 80 =$
$\cos 20 = \sin 70$
$\cos 30 = \sin 60 = \sqrt{3}/2$
$\cos 45 = \sin 45 = \frac{1}{\sqrt{2}}$
$\cos 60 = \sin 30 = \frac{1}{2}$
$\cos 90 = \sin 0 = 0$

Identity of the Trigonometric Functions :-

هوية الدوال المثلثية

① Double Angle Identity :-

* $\sin 2\theta = 2\sin\theta\cos\theta$
* $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
* $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

② Half Angle Identity :-

$$\sin\frac{\theta}{2} = \mp\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \mp\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

EX) proof that $\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$

[solution]

From the half angle identity,

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} * \sqrt{\frac{1-\cos\theta}{1-\cos\theta}} = \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}$$

From eqn ① $\Longrightarrow \sin^2\theta + \cos^2\theta = 1 \rightarrow 1-\cos^2\theta = \sin^2\theta$

$$\therefore \tan\frac{\theta}{2} = \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} = \frac{1-\cos\theta}{\sin\theta} \qquad \text{o.k}$$

EX) proof that $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$

[solution]

From the half angle identity,

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} * \sqrt{\frac{1+\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\cos^2\theta}{(1+\cos\theta)^2}}$$

From previous ex. $\Longrightarrow 1-\cos^2\theta = \sin^2\theta$

$$\therefore \tan\frac{\theta}{2} = \sqrt{\frac{\sin^2\theta}{(1+\cos\theta)^2}} = \frac{\sin\theta}{1+\cos\theta} \qquad \text{o.k}$$

③ Sum & Difference Identity :- هويات الجمع والطرح

* $\sin(\alpha \mp \beta) = \sin\alpha\cos\beta \mp \cos\alpha\sin\beta$
* $\cos(\alpha \mp \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$
* $\tan(\alpha \mp \beta) = \frac{\tan\alpha \mp \tan\beta}{1 \pm \tan\alpha\tan\beta}$

i.e; $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

(4) The Power Reducing Formulas: علاقات تخفيض القوة (الاس)

* $\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$
* $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$
* $\tan^2\theta = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$

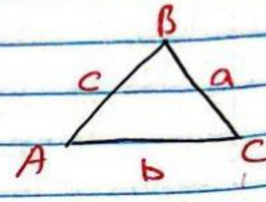
(5) The Product To Sum Formulas: علاقات الضرب إلى الجمع

* $\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$
* $\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$
* $\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$
* $\cos\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$

(6) Sum to Product Formulas: علاقات الجمع إلى الضرب

* $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
* $\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$
* $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
* $\cos\alpha - \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

(7) Law of Sines :- قانون الجيوب



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

(8) Law of Cosines :- قانون الجيوب تمام

$$c^2 = a^2 + b^2 - 2ab \cos C$$

To calalate the area of the triangle

$$Area = A = \frac{1}{2} ab \sin C$$

or

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{a+b+c}{2}$$

(9) Law of Tangents :- قانون الظلال

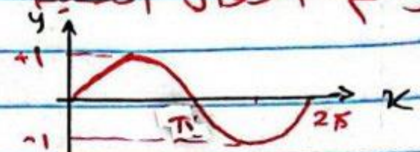
It's no longer using, due to its complication;

$$\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$$

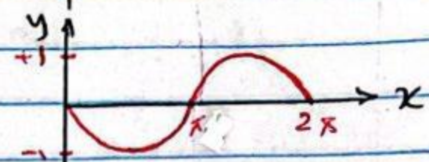
Graphing the Trigonometric Functions :-

رسم الدوال المثلثية

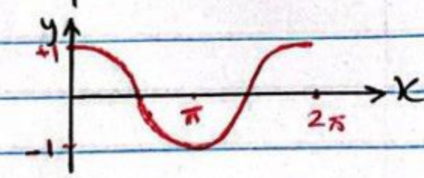
① $y = +\sin x \longrightarrow$



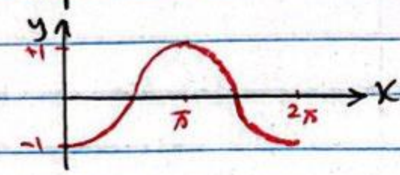
② $y = -\sin x \longrightarrow$



③ $y = +\cos x \longrightarrow$



④ $y = -\cos x \longrightarrow$

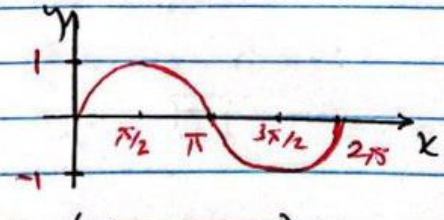


Note)

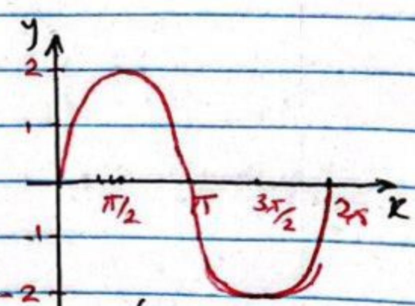
From $0 \sim 2\pi$ "one cycle"

Q: what is the difference between
$y = \sin x$ and,
$y = 2\sin x$

Answer



$(y = \sin x)$



$(y = 2\sin x)$

* The amplitude of $y = 2\sin x$ is twice that of $y = \sin x$.

The general form of a sine wave is;

$$y = A\sin(BX + C) + D$$

A --- Amplitude
B --- used to calculate the period
C --- phase shifting
D --- vertical shifting

$$\text{period} = P = \frac{2\pi}{B}$$

- If $D = 3 \longrightarrow$ this means shift 3 units up.
- If $D = -2 \Longrightarrow$ " " " 2 " down.

To calculate a phase shift $\rightarrow$ $BX + C = 0$

$\therefore X = \frac{-C}{B}$

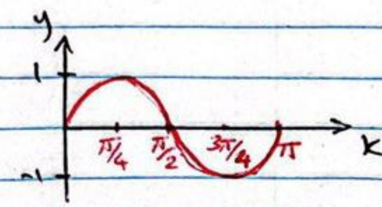
Ex) Graph $y = \sin 2x$?

Answer

Amplitude $= A = \boxed{1}$,

Period $= \frac{2\pi}{B}$, $B = 2$

$\therefore$ period $= \frac{2\pi}{2} = \boxed{\pi}$



Range $\rightarrow [-1, 1]$

Ex) Graph $y = \sin x + 1$

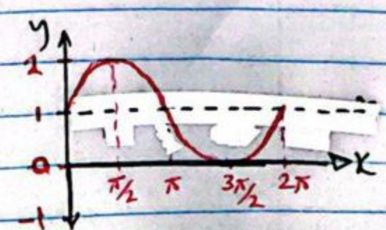
Answer

- Amplitude $= A = \boxed{1}$
- period $= P = \frac{2\pi}{B}$, $B = 1$

$\therefore P = \boxed{2\pi}$

- vertical shifting $= D = \boxed{1}$

(1 unit shifting up)

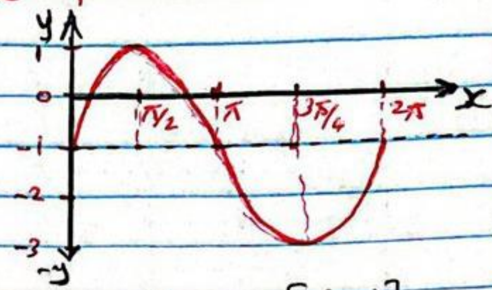


Range $\rightarrow [0, 2]$

Ex) Graph $y = 2\sin x - 1$

Answer

- Amplitude $= A = \boxed{2}$
- Period $= \frac{2\pi}{B}$, $B = 1$
- $\therefore$ period $= \boxed{2\pi}$
- vertical shifting $= D = \boxed{-1}$

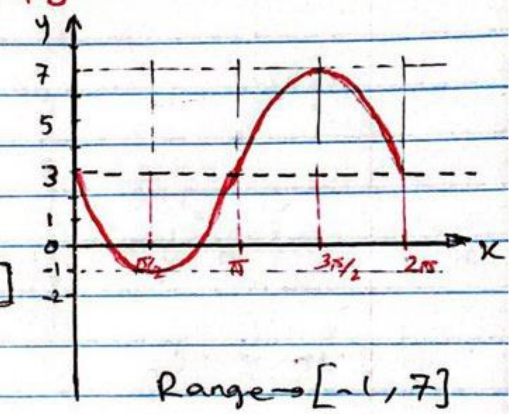


Range $\to [-3, 1]$

Ex) Graph $y = -4\sin x + 3$

Answer

- Amplitude $= A = \boxed{4}$
- period $= \frac{2\pi}{B}$, $B = 1$
- $\therefore$ period $= \boxed{2\pi}$
- Vertical shifting $= D = \boxed{3}$

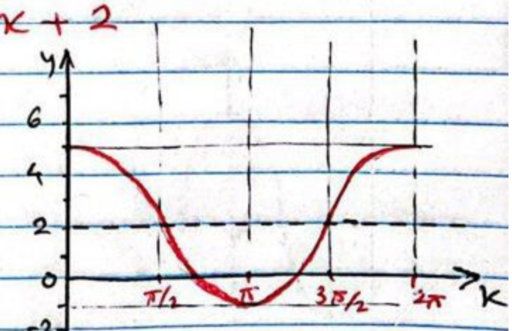


Range $\to [-1, 7]$

Ex) Graph $y = 3\cos x + 2$

Answer

- Amplitude $= A = \boxed{3}$
- Period $= \frac{2\pi}{B}$, $B = 1$
- $\therefore$ period $= \boxed{2\pi}$
- Vertical shifting $= D = 2$

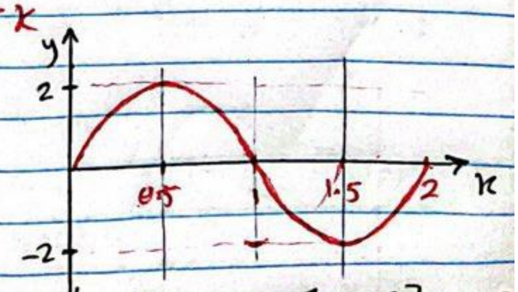


Range $\to [-1, 5]$

Ex) Graph $y = 2\sin \pi x$

Answer

- Amplitude $= \boxed{2}$
- Period $= P = \frac{2\pi}{B} = \frac{2\pi}{\pi} = \boxed{2}$
- No, vertical shifting

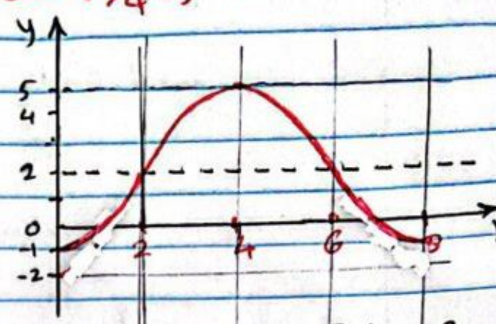


Range $\to [-2, 2]$

Ex) Graph $y = -3\cos(\frac{\pi}{4}x) + 2$

Answer

- Amplitude = $\boxed{3}$
- period = $\frac{2\pi}{B}$, $B = \pi/4$
- $\therefore$ period = $\frac{2\pi}{\pi/4} = \boxed{8}$
- Vertical shifting = D = $\boxed{2}$

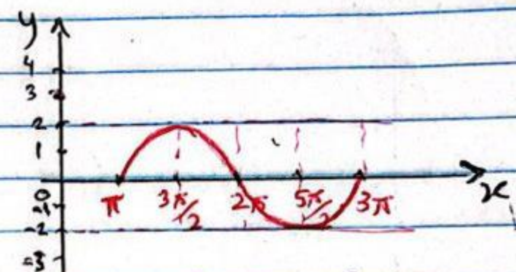


Range → [-1, 5]

Ex) Graph $y = 2\sin(x - \pi)$

Answer

- Amplitude = A = $\boxed{2}$
- period = $\frac{2\pi}{B}$, B = 1
- $\therefore$ period = $\boxed{2\pi}$
- No Vertical shifting , D = 0
- phase shifting :
  $x - \pi = 0 \longrightarrow \boxed{x = \pi}$

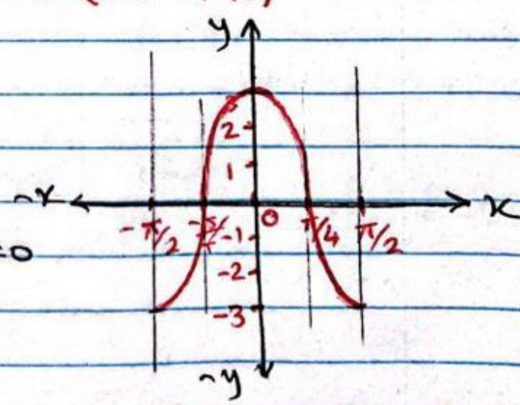


Range → [-2, 2]

Ex) Graph $y = -3\cos(2x + \pi)$

Answer

- Amplitude = A = $\boxed{3}$
- period = $\frac{2\pi}{B}$, B = 2
- $\therefore$ period = $\boxed{\pi}$
- No vertical shifting , D = 0
- Phase shifting
  $2x + \pi = 0 \longrightarrow \boxed{x = -\frac{\pi}{2}}$

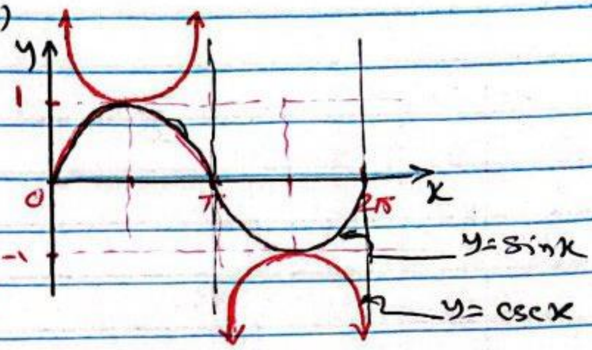


Range → [-3, 3]

Ex) Graph $y = \csc x = \frac{1}{\sin x}$

Answer

- It's same for $y = \sin x$, but here we have asymptotes (محاذيات)



* The domain of csc x is all x except

$x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \ldots$

or $x \neq n\pi$, $n = 0, \mp 1, \mp 2, \ldots$

Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

$\cup$ --- Union

Ex) Graph $y = \sec x = \frac{1}{\cos x}$

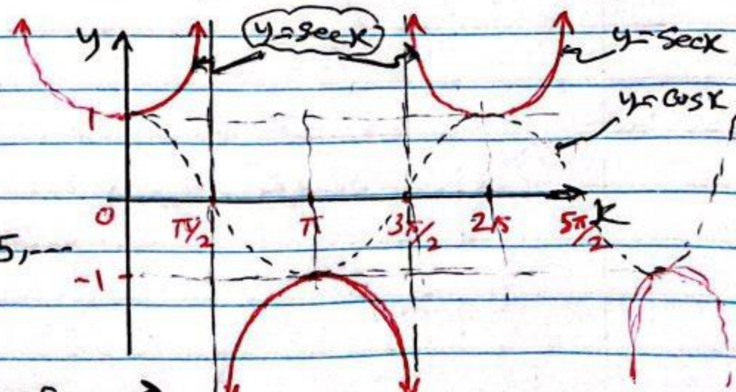
Answer

- Here we need to graph cos x first, then we can graph sec x, similar to what we did in previous example.

Domain

$x \neq n\frac{\pi}{2}$,

$n = \mp 1, \mp 3, \mp 5, \ldots$



Range $\rightarrow (-\infty, -1] \cup [1, \infty)$

Ex) Graph $y = \tan x$

Answer

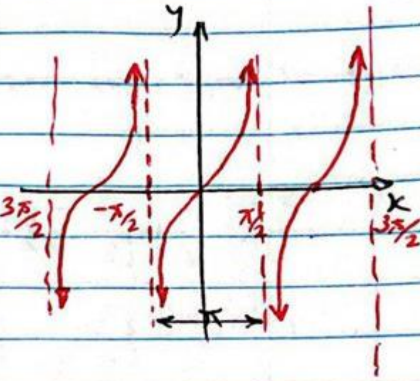
The period of tan is different from sine & Cosine's once ; it is,

$P = \frac{\pi}{B} \Longrightarrow P = \frac{\pi}{1} = \boxed{\pi}$

Range $\rightarrow (-\infty, \infty)$

Domain $\rightarrow x \neq n\frac{\pi}{2}$.

$n = \mp 1, \mp 3, \ldots$



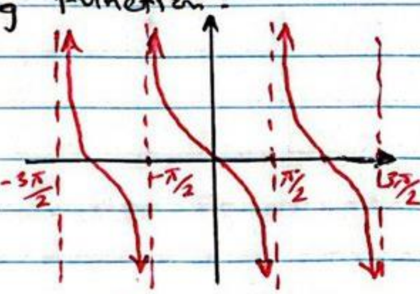
Ex) Graph $y = -\tan x$

Answer

It is exactly, similar to the previous example, but $-$ve tan fun is decreasing function.

- Period $= P = \frac{\pi}{B} = \boxed{\pi}$
- Range $\rightarrow (-\infty, \infty)$
- Domain $\rightarrow x \neq n\frac{\pi}{2}$

$n = \mp 1, \mp 3, \ldots$



⊛ Note)

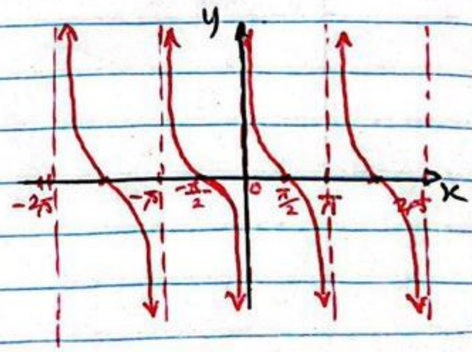
To graph $\cot x$, it is similar to $-\tan x$, & to graph $-\cot x$, it is similar to $\tan x$.

Ex) Graph $y = \cot x$

Answer

- period $= P = \frac{\pi}{B} = \boxed{\pi}$
- Range $\rightarrow (-\infty, \infty)$
- Domain $\rightarrow x \neq n\pi$

$n = 0, \mp 1, \mp 2, \ldots$



EX) Graph $y = -2\cot(\frac{1}{2}x - \pi) + 3$

Answer

- First of all, we need to find the period,

Period $= P = \frac{\pi}{B}$, $B = \frac{1}{2}$

$\therefore P = \frac{\pi}{1/2} = \boxed{2\pi}$

- To find the vertical asymptote, we need to set

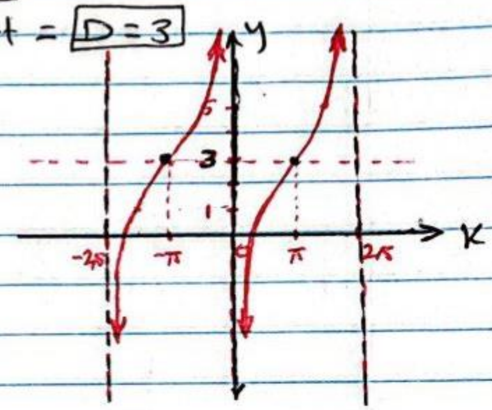
$\frac{1}{2}x - \pi = 0 \longrightarrow \boxed{x = 2\pi}$

- We have a vertical shift $= \boxed{D = 3}$

Range $\to (-\infty, \infty)$

Domain $\to x \neq n\pi$

$n = 0, \mp 2, \mp 4, \mp 6, \ldots$



H.W #3 :

Graph the following Trigonometric Functions:

1- $y = 2\csc(\frac{\pi}{4}x) + 1$

2- $y = -2\sec(2x - \pi) + 6$

3- $y = 3\tan x + 2$

4- $y = -2\tan(\frac{1}{4}x - \pi) + 3$

---- نهاية محاضرة " Trigonometric Functions & Relations, Trigonometric Equations, Graphing of Functions الدوال المثلثية وعلاقاتها، المعادلات المثلثية، رسم الدوال المثلثية" ----