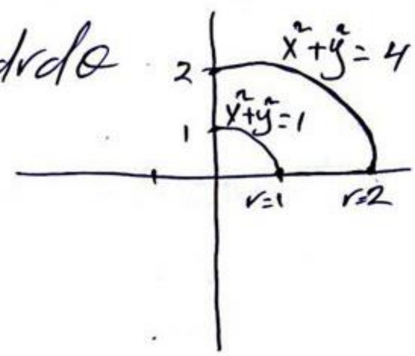
Double Integrals in Polar Coordinates

Ex1| Evaluate the integral $\iint_R 3x\,dA$ over the region $R = \{(r,\theta),\ 1 \le r \le 2,\ 0 \le \theta \le \pi/2\}$

sol|

$$\iint_R 3x\,dA = \int_{\theta=0}^{\theta=\pi/2}\int_{r=1}^{r=2} 3r\cos\theta\, r\,dr\,d\theta$$



$$= \int_0^{\pi/2} \frac{3r^3}{3}\cos\theta\Big]_1^2\,d\theta$$

$$= \int_0^{\pi/2} 7\cos\theta\,d\theta$$

$$= 7\sin\theta\Big]_0^{\pi/2}$$

$$= 7(1) - 7(0) = 7$$

H.W | sketch the region $D = \{(r,\theta),\ 1 \le r \le 2,\ -\pi/2 \le \theta \le \pi/2\}$ and evaluate $\iint_R x\,dA$

Ex2) Evaluating a Double Integral by converting from Cartesian coordinates $\iint_R (1-x^2-y^2)\,dy$ where R is the unit circle on the xy-plane.

Sol)

the region R is a unit circle, so we can describe it as $R = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$$\iint_R (1-x^2-y^2)\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}$$

Ex 3) Evaluating a Double Integral by converting from Cartesian coordinates $\iint_R (x+y)\,dA$

where $R = \{(x,y) \mid 1 \leq x^2+y^2 \leq 4,\ x \leq 0\}$

sol)

we can see that R is an annular region that can be converted to Polar coordinates and described as $R = \{(r,\theta) \mid 1 \leqslant r \leqslant 2,\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\}$

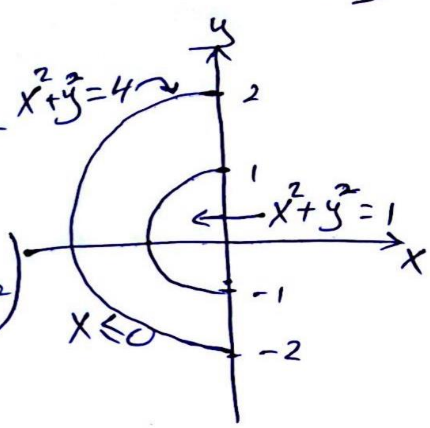
$$\iint_R (x+y)\,dA$$

$$= \int_{\pi/2}^{3\pi/2} \int_1^2 (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= \left(\int_1^2 r^2\,dr\right)\left(\int_{\pi/2}^{3\pi/2} (\cos\theta+\sin\theta)\,d\theta\right)$$

$$= \frac{r^3}{3}\Big]_1^2 \ \sin\theta - \cos\theta\Big]_{\pi/2}^{3\pi/2}$$

$$= -\frac{14}{3}$$



H.W.) Evaluate the integral

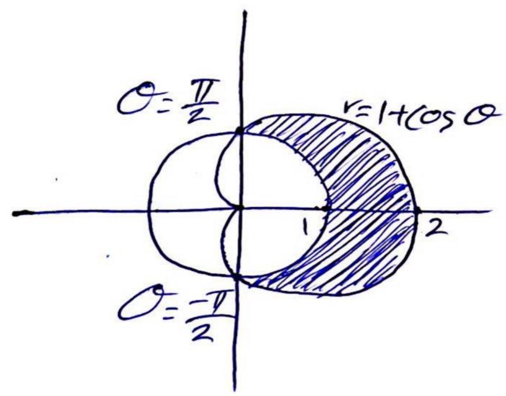
$$\iint_R (4-x^2-y^2)\,dy$$

where R is the circle of radius 2 on the xy-Plane

Ex 4 | Find the limits of integration for integrating $f(r,\theta)$ over the region R that lies inside the cardioid $r=1+\cos\theta$ and outsid the circle $r=1$

sol |

(1) sketch the region



(2) find the r-limits

$r_1=1 \quad , r=1+\cos\theta$

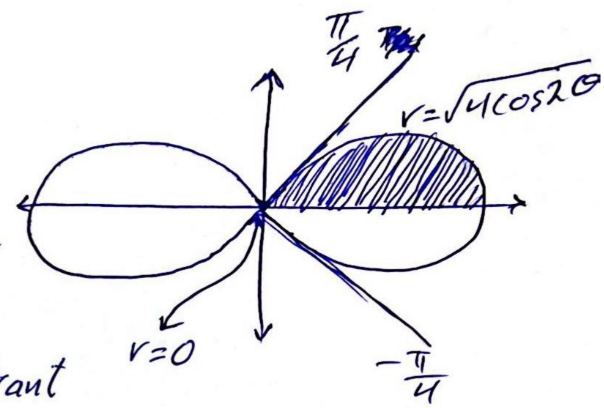
(3) find the θ-limits

$\theta=-\pi/2$ to $\theta=\pi/2$

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r,\theta)\,r\,dr\,d\theta$$

Ex 5| Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$

Sol|



from the symmetry of regine we see that the total area is 4 times the first-quadrant portion

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\, d\theta$$

$$= 4\sin 2\theta \Big]_0^{\pi/4}$$

$$= 4$$