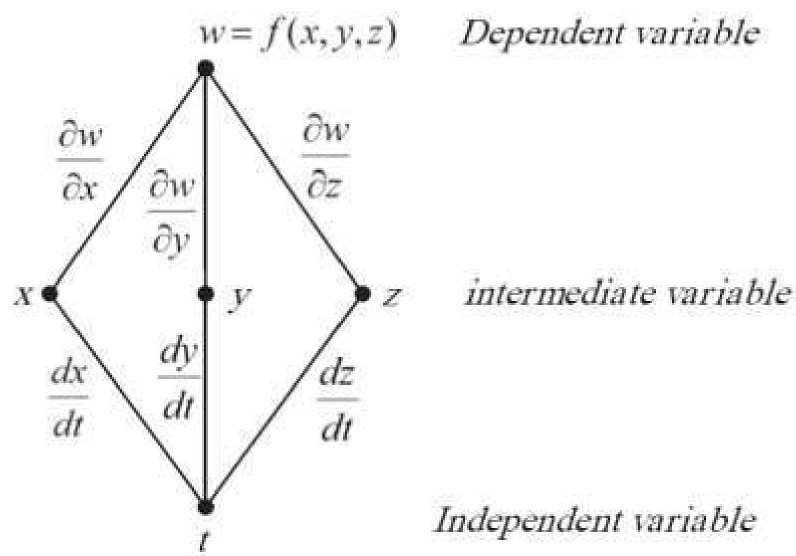
Partial Derivatives

The chain rule for function of three variables:

If $w = f(x, y, z)$ is differentiable and x , y and z are differentiable function of t , then w is a differentiable function of t and:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

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Example: Use the chain rule to find the derivative $\left(\frac{dw}{dt}\right)$ of $w = xy + z$ with $x = \cos t$, $y = \sin t$, $z = t$, and determine the value of $\left(\frac{dw}{dt}\right)$ at $t = 0$

Solution:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial}{\partial x}(xy + z).\frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(xy + z).\frac{d}{dt}(\sin t) + \frac{\partial}{\partial z}(xy + z).\frac{d}{dt}(t)$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$= \cos 2t + 1$$

$$\left(\frac{dw}{dt}\right)_{t=0} = \cos(0) + 1 = 2$$

H.W: Use the chain rule to find the derivative $\left(\frac{dw}{dt}\right)$ of $w = \frac{x}{z} + \frac{y}{z}$ with $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$, and determine the value of $\left(\frac{dw}{dt}\right)$ at $t = 3$

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Directional derivatives and gradient vectors:

Gradient vector:

The gradient vector (gradient) of $f(x,y,z)$ at a point $P_o(x_o, y_o, z_o)$ is the vector:

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

obtained by evaluating the partial derivatives of f at P_o

The notation ∇f is read (grad f) as well as (gradient f) and (del f).
The symbol ∇ by itself is read (del)

The directional derivatives

If $f(x,y,z)$ has continuous partial derivatives at $P_o(x_o, y_o, z_o)$ and u is a unit vector, then **the derivative of f at P_o in the direction of u is**:

$$\left(\frac{df}{ds}\right)_{u,P_o} = (\nabla f)_{P_o} . u$$

Which is the scalar product of the gradient of f at P_o and u

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Example: find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_o(1,1,0)$ in the direction of vector $A = 2i - 3j + 6k$

Solution:

$$u = \frac{A}{|A|}$$

$$|A| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$u = \frac{A}{|A|} = \frac{2i - 3j + 6k}{7} = \frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k$$

The partial derivatives of f at P_o are

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 - xy^2 - z) = \boxed{3x^2 - y^2}$$

$$\therefore\ f_x(1,1,0) = (3)(1)^2 - (1)^2 = 3 - 1 = \boxed{2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 - xy^2 - z) = \boxed{-2xy}$$

$$\therefore\ f_y(1,1,0) = -(2)(1)(1) = \boxed{-2}$$

$$f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^3 - xy^2 - z) = \boxed{-1}$$

$$f_z(1,1,0) = \boxed{-1}$$

The gradient of f at P_o is:

$$\nabla f\big|_{(1,1,0)} = f_x(1,1,0)i + f_y(1,1,0)j + f_z(1,1,0)k = 2i - 2j - k$$

The derivative of f at P_o in the direction A is therefore:

$$(D_u f)\big|_{(1,1,0)} = \nabla f\big|_{(1,1,,0)}.u$$

$$= (2i - 2j - k).\left(\frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}k\right)$$

$$= (2)\left(\frac{2}{7}\right) + (-2)\left(-\frac{3}{7}\right) + (-1)\left(\frac{6}{7}\right) = \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \boxed{\frac{4}{7}}$$

Example: find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $v = 3i - 4j$

Solution: the direction of v is the unit vector obtained by dividing v by its length:

$$u = \frac{v}{|v|}$$

$$|v| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$u = \frac{3i - 4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xe^y + \cos(xy)) = \boxed{e^y - y\sin(xy)}$$

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$$\therefore\ f_x(2,0) = e^0 - (0)\sin(2)(0)) = e^0 - 0 = \boxed{1}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xe^y + \cos(xy)) = \boxed{xe^y - x\sin(xy)}$$

$$\therefore\ f_y(2,0) = (2)(e^0) - (2)\sin(2)(0)) = 2e^0 - (2)(0) = (2)(1) - 0 = \boxed{2}$$

The gradient of f at (2 , 0) is:

$$\nabla f\big|_{(2,0)} = f_x(2,0)i + f_y(2,0)j = i + 2j$$

The derivative of f at (2 , 0) in the direction v is therefore:

$$(D_u f)\big|_{(2,0)} = \nabla f\big|_{(2,0)}.u$$

$$= (i + 2j).\left(\frac{3}{5}i - \frac{4}{5}j\right)$$

$$= (1)\left(\frac{3}{5}\right) + (2)\left(-\frac{4}{5}\right)$$

$$= \frac{3}{5} - \frac{8}{5} = \frac{-5}{5} = \boxed{-1}$$

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Example: find the derivative of function $f(x,y) = 2xy - 3y^2$ at the point P_o (5 , 5) in the direction of $A = 4i + 3j$

Solution:

$$u = \frac{A}{|A|}$$

$$|A| = \sqrt{(4)^2 + (3)^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$u = \frac{4i+3j}{5} = \frac{4}{5}i + \frac{3}{5}j$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2xy - 3y^2) = \boxed{2y}$$

$$\therefore\ f_x(5,5) = (2)(5) = \boxed{10}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2xy - 3y^2) = \boxed{2x - 6y}$$

$$\therefore\ f_y(5,5) = (2)(5) - (6)(5) = 10 - 30 = \boxed{-20}$$

The gradient of f at (5 , 5) is:

$$\nabla f\big|_{(5,5)} = f_x(5,5)i + f_y(5,5)j = 10i - 20j$$

The derivative of f at (5 , 5) in the direction A is therefore:

$$(D_u f)\big|_{(5,5)} = \nabla f\big|_{(5,5)} . u$$

$$=(10i-20j).\left(\frac{4}{5}i+\frac{3}{5}j\right)$$

$$=(10)\left(\frac{4}{5}\right)+(-20)\left(\frac{3}{5}\right)$$

$$=\frac{40}{5}-\frac{60}{5}=\frac{-20}{5}=\boxed{-4}$$

H.W:

1. find the derivative of the function $f(x,y,z)=xy+yz+zx$,at the point $P_o(1,-1,2)$ in the direction of $A=3i+6j-2k$

2. find the derivative of the function $g(x,y,z)=3e^x\cos yz$,at the point $P_o(0,0,0)$ in the direction of $A=2i+j-2k$

Tangent planes and normal lines:

The **tangent plane** at the point $P_o(x_o,y_o,z_o)$ on the level surface $f(x,y,z)=C$ of a differentiable function f is the plane through P_o normal to $\nabla f|_{P_o}$

The **normal line** of the surface at P_o is the line through P_o parallel to $\nabla f|_{P_o}$

The tangent plane and normal line have the following equation:

Tangent plane to $f(x,y,z)=C$ at $P_o(x_o,y_o,z_o)$:

$$f_x(p_o)(x-x_o)+f_y(p_o)(y-y_o)+f_z(p_o)(z-z_o)=0$$

Normal line to $f(x,y,z)=C$ at $P_o(x_o,y_o,z_o)$:

$$x=x_o+f_x(p_o)t\ ,\quad y=y_o+f_y(p_o)t\ ,\quad z=z_o+f_z(p_o)t$$

Example: find the tangent plane and normal line of the surface $f(x,y,z)=x^2+y^2+z-9=0$ at the point $P_o(1,2,4)$

Solution: the tangent plane is:

$$f_x(p_o)(x-x_o)+f_y(p_o)(y-y_o)+f_z(p_o)(z-z_o)=0$$

$$f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z-9)=2x$$

$$f_x(P_o)=f_x(1,2,4)=(2)(1)=\boxed{2}$$

$$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2+z-9)=2y$$

$$f_y(P_o)=f_y(1,2,4)=(2)(2)=\boxed{4}$$

$$f_z=\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}(x^2+y^2+z-9)=1$$

$$f_z(P_o)=f_z(1,2,4)=\boxed{1}$$

$\therefore$ The tangent plane is:

$$\boxed{2(x-1)+4(y-2)+(z-4)=0}$$

or

$$2x-2+4y-8+z-4=0$$
$$2x+4y+z-14=0$$
$$2x+4y+z=14$$

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The normal line is:

$$x = x_o + f_x(p_o)t \quad , \quad y = y_o + f_y(p_o)t \quad , \quad z = z_o + f_z(p_o)t$$

$$\therefore \quad x = 1 + 2t \quad , \quad y = 2 + 4t \quad , \quad z = 4 + t$$

Example: find the tangent plane and normal line of the surface $f(x, y, z) = x^2 + y^2 + z^2 = 3$ at the point $P_o(1,1,1)$

Solution: the tangent plane is:

$$f_x(p_o)(x - x_o) + f_y(p_o)(y - y_o) + f_z(p_o)(z - z_o) = 0$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2x$$

$$f_x(P_o) = f_x(1,1,1) = (2)(1) = \boxed{2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + z^2) = 2y$$

$$f_y(P_o) = f_y(1,1,1) = (2)(1) = \boxed{2}$$

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$$f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2 + y^2 + z^2) = 2z$$

$$f_z(P_o) = f_z(1,1,1) = (2)(1) = \boxed{2}$$

$\therefore$ The tangent plane is:

$$\boxed{2(x-1) + 2(y-1) + 2(z-1) = 0}$$

or

$$2x - 2 + 2y - 2 + 2z - 2 = 0$$
$$2x + 2y + 2z - 6 = 0$$
$$2x + 2y + 2z = 6$$
$$2(x + y + z) = 6 \implies x + y + z = 3$$

The normal line is:

$$x = x_o + f_x(p_o)t \quad , \quad y = y_o + f_y(p_o)t \quad , \quad z = z_o + f_z(p_o)t$$

$$\therefore \quad x = 1 + 2t \quad , \quad y = 1 + 2t \quad , \quad z = 1 + 2t$$

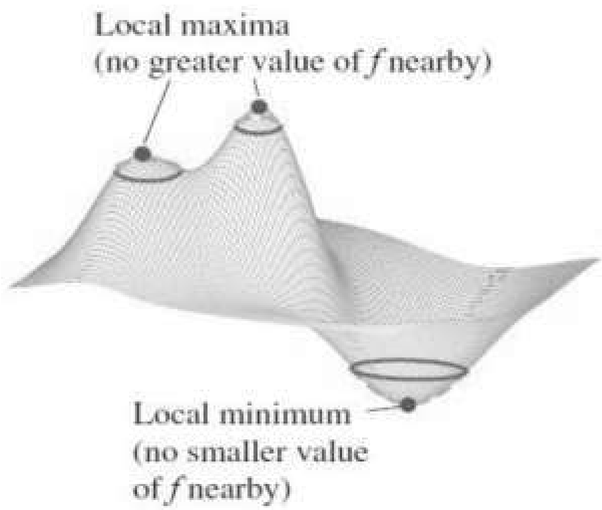
H.W: find the tangent plane and normal line of the surface $f(x, y, z) = x^2 + 2xy - y^2 + z^2 = 7$ at the point $P_o(1,-1,3)$

Extreme values and saddle points:

Derivative test:

To find the local extreme values of a function of a single, we look for points where the graph has a horizontal tangent line. At such points, we then look for **local maxima**, **local minima**, and points of inflection. For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a horizontal tangent plane. At such points, we then look for **local maxima**, **local minima**, and **saddle points** (more about saddle points in a moment). Local maxima correspond to mountain peak on the surface $z = f(x, y)$ and local minima correspond to valley bottoms. At such points the tangent planes, when they exist are horizontal.

Local extreme are also called ***relative extreme***.



Critical point: an interior point of the domain a function $f(x, y)$ where both f_x and f_y are zero or where one or both f_x and f_y do not exist is *a critical point* of f

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Saddle point: a differentiable function $f(x,y)$ has a saddle point at critical point (a,b) if in every open disk centered at (a,b) there are domain points (x,y) where $f(x,y) > f(a,b)$ and domain points (x,y) where $f(x,y) < f(a,b)$ the corresponding point $(a,b,f(a,b))$ on the Surface $z = f(x,y)$ is called *a saddle point* of the surface

***Second derivative test for local extreme values*:**
Suppose that $f(x,y)$ and its first and second partial derivative are continuous throughout a disk centered at (a,b) and that:

$f_x = 0$ and $f_y = 0 \Longrightarrow$ solve these equation to find the value of $(x,y) = (a,b) \Longrightarrow$ (*critical point*)

Then:

1. if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a,b) \Longrightarrow$ then f has **a local maximum** at (a,b)

2. if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a,b) \Longrightarrow$ then f has **a local minimum** at (a,b)

3. if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a,b) \Longrightarrow$ then f has **a saddle point** at (a,b)

4. if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a,b) \Longrightarrow$ then **the test inconclusive** at (a,b).
In this case we must find some other way to determine the behavior of f at (a,b)

Example: find the extreme values of the function

$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy - x^2 - y^2 - 2x - 2y + 4) = \boxed{y - 2x - 2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy - x^2 - y^2 - 2x - 2y + 4) = \boxed{x - 2y - 2}$$

$$\left.\begin{array}{l} f_x = 0 \Longrightarrow y - 2x - 2 = 0 \\ f_y = 0 \Longrightarrow x - 2y - 2 = 0 \end{array}\right\} \Longrightarrow$$ Solve these equation to find $(x, y) \Longrightarrow (a, b)$

$$\left.\begin{array}{l} x = -2 \Longrightarrow a = -2 \\ y = -2 \Longrightarrow b = -2 \end{array}\right\} \Longrightarrow$$ Critical point $(-2,-2)$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(y - 2x - 2) = -2$$

$$\therefore\ f_{xx}(-2,-2) = \boxed{-2}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x - 2y - 2) = -2$$

$$\therefore\ f_{yy}(-2,-2) = \boxed{-2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(y - 2x - 2) = 1$$

$$\therefore\ f_{xy}(-2,-2) = \boxed{1}$$

$$f_{xx}f_{yy} - f_{xy}^{\ 2} = (-2)(-2) - (1)^2 = 4 - 1 = \boxed{3}$$

$f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^{\ 2} > 0 \Longrightarrow \therefore f$ has a local maximum at $(-2,-2)$

The value of *f* at this point is:

$$f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - (2)(-2) - (2)(-2) + 4$$
$$= 4 - 4 - 4 + 4 + 4 + 4 = \boxed{8}$$

Example: find the local maxima, local minima, and saddle point of the function

$$f(x, y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$$

Solution:

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + 3y^2 - 6x + 3y - 6) = \boxed{2x + 3y - 6}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + 3y^2 - 6x + 3y - 6) = \boxed{3x + 6y + 3}$$

$$\left.\begin{array}{l} f_x = 0 \Longrightarrow 2x + 3y - 6 = 0 \\ f_y = 0 \Longrightarrow 3x + 6y + 3 = 0 \end{array}\right\} \Longrightarrow$$ Solve these equation to find $(x, y) \Longrightarrow (a, b)$

$$\left.\begin{array}{l} x = 15 \Longrightarrow a = 15 \\ y = -8 \Longrightarrow b = -8 \end{array}\right\} \Longrightarrow$$ Critical point $(15,-8)$

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$$f_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})=\frac{\partial}{\partial x}(2x+3y-6)=2$$

$$\therefore\ f_{xx}(15,-8)=\boxed{2}$$

$$f_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}(\frac{\partial f}{\partial y})=\frac{\partial}{\partial y}(3x+6y+3)=6$$

$$\therefore\ f_{yy}(15,-8)=\boxed{6}$$

$$f_{xy}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})=\frac{\partial}{\partial y}(2x+3y-6)=3$$

$$\therefore\ f_{xy}(15,-8)=\boxed{3}$$

$$f_{xx}f_{yy}-f_{xy}{}^2=(2)(6)-(3)^2=12-9=\boxed{3}$$

$f_{xx}>0$ and $f_{xx}f_{yy}-f_{xy}{}^2>0 \implies \therefore f$ has a local minimum at $(15,-8)$

The value of f at this point is:

$$f(15,-8)=(15)^2+(3(15)(-8)+(3)(-8)^2-(6)(15)+(3)(-8)-6$$
$$=225-360+192-90-24-6=\boxed{-63}$$

***H.W*:** find the local maxima, local minima, and saddle point of the functions:

1. $f(x,y)=x^2+xy+y^2+3x-3y+4$

2. $f(x,y)=x^2+xy+3x+2y+5$

3. $f(x,y)=2x^2+3xy+4y^2-5x+2y$

4. $f(x,y)=2xy-x^2-2y^2+3x+4$

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