Double Integrals in Polar Coordinates

To this point we've seen quite a few double integrals. However, in every case we've seen to this point the region DD could be easily described in terms of simple functions in Cartesian coordinates. In this section we want to look at some regions that are much easier to describe in terms of polar coordinates. For instance, we might have a region that is a disk, ring, or a portion of a disk or ring. In these cases, using Cartesian coordinates could be somewhat cumbersome. For instance, let's suppose we wanted to do the following integral,

$$\iint_D f(x,y)\,dA, \quad D \text{ is the disk of radius 2}$$

To this we would have to determine a set of inequalities for x and y that describe this region. These would be,

$$-2 \le x \le 2$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$$

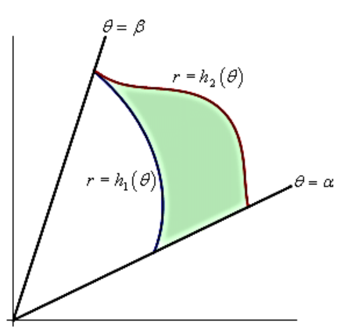
With these limits the integral would become,

$$\iint_D f(x,y)\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

These are very simple limits and, in fact, are constant limits of integration which almost always makes integrals somewhat easier.

So, if we could convert our double integral formula into one involving polar coordinates we would be in pretty good shape. The problem is that we can't just convert the dx and the dy into a dr and a dθ. In computing double integrals to this point we have been using the fact that dA=dxdy and this really does require Cartesian coordinates to use. Once we've moved into polar coordinates dA≠drdθ and so we're going to need to determine just what dA is under polar coordinates.

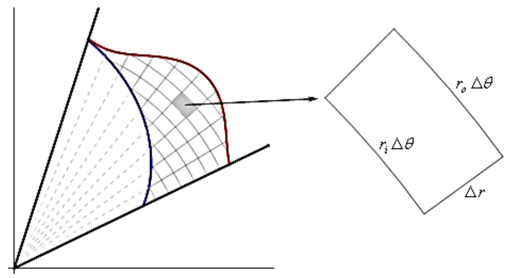
So, let's step back a little bit and start off with a general region in terms of polar coordinates and see what we can do with that. Here is a sketch of some region using polar coordinates.



So, our general region will be defined by inequalities,

$$\alpha \leq \theta \leq \beta$$
$$h_1(\theta) \leq r \leq h_2(\theta)$$

Now, to find dA let's redo the figure above as follows,



As shown, we'll break up the region into a mesh of radial lines and arcs. Now, if we pull one of the pieces of the mesh out as shown we have something that is almost, but not quite a rectangle. The area of this piece is ΔA. The two sides of this piece both have length Δr=ro−ri where ro is the radius of the outer arc and ri is the radius of the inner arc. Basic geometry then tells us that the length of the inner edge is riΔθ while the length of the out edge is roΔθ where Δθ is the angle between the two radial lines that form the sides of this piece.

Now, let's assume that we've taken the mesh so small that we can assume that ri≈ro=r and with this assumption we can also assume that our piece is close enough to a rectangle that we can also then assume that,

$$\Delta A \approx r\,\Delta\theta\,\Delta r$$

Also, if we assume that the mesh is small enough then we can also assume that,

$$dA \approx \Delta A \qquad d\theta \approx \Delta\theta \qquad dr \approx \Delta r$$

With these assumptions we then get dA≈rdrdθ.

In order to arrive at this we had to make the assumption that the mesh was very small. This is not an unreasonable assumption. Recall that the definition of a double integral is in terms of two limits and as limits go to infinity the mesh size of the region will get smaller and smaller. In fact, as the mesh size gets smaller and smaller the formula above becomes more and more accurate and so we can say that,

$$dA = r\,dr\,d\theta$$

We'll see another way of deriving this once we reach the Change of Variables section later in this chapter. This second way will not involve any assumptions either and so it maybe a little better way of deriving this.

Before moving on it is again important to note that dA≠drdθ. The actual formula for dA has an r in it. It will be easy to forget this rr on occasion, but as you'll see without it some integrals will not be possible to do.

Now, if we're going to be converting an integral in Cartesian coordinates into an integral in polar coordinates we are going to have to make sure that we've also converted all the x's and y's into polar coordinates as well. To do this we'll need to remember the following conversion formulas,

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r^2 = x^2 + y^2$$

We are now ready to write down a formula for the double integral in terms of polar coordinates.

$$\iint\limits_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

It is important to not forget the added r and don't forget to convert the Cartesian coordinates in the function over to polar coordinates.

Let's look at examples of these kinds of integrals.

Example 1 Evaluate the following integrals by converting them into polar coordinates.

∬D2xydA, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

(a) $\iint\limits_D 2x\,y\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.

First let's get D in terms of polar coordinates. The circle of radius 2 is given by $r = 2$ and the circle of radius 5 is given by $r = 5$. We want the region between the two circles, so we will have the following inequality for r.

$$2 \le r \le 5$$

Also, since we only want the portion that is in the first quadrant we get the following range of θ's.

$$0 \le \theta \le \frac{\pi}{2}$$

Now that we've got these we can do the integral.

$$\iint\limits_D 2x\,y\,dA = \int_0^{\frac{\pi}{2}} \int_2^5 2\,(r\cos\theta)\,(r\sin\theta)\,r\,dr\,d\theta$$

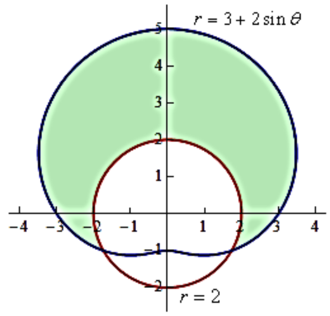
Don't forget to do the conversions and to add in the extra r. Now, let's simplify and make use of the double angle formula for sine to make the integral a little easier.

$$\begin{aligned}\iint\limits_D 2x\,y\,dA &= \int_0^{\frac{\pi}{2}} \int_2^5 r^3 \sin(2\theta)\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \left.\frac{1}{4} r^4 \sin(2\theta)\right|_2^5 d\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{609}{4}\sin(2\theta)\,d\theta \\ &= \left.-\frac{609}{8}\cos(2\theta)\right|_0^{\frac{\pi}{2}} \\ &= \frac{609}{4}\end{aligned}$$

Example 2 Determine the area of the region that lies inside r=3+2sinθ and outside r=2.

Solution;

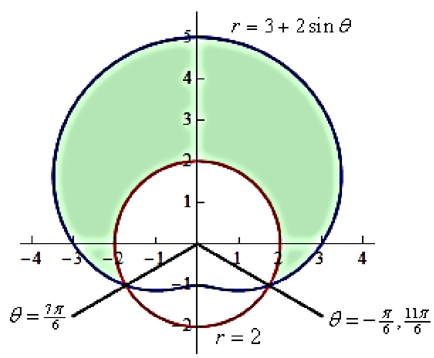
Here is a sketch of the region, D, that we want to determine the area of.



To determine this area we'll need to know that value of θ for which the two curves intersect. We can determine these points by setting the two equations equal and solving.

$$3 + 2\sin\theta = 2$$
$$\sin\theta = -\frac{1}{2} \qquad \Rightarrow \qquad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Here is a sketch of the figure with these angles added.



Note as well that we've acknowledged that $-\frac{\pi}{6}$ is another representation for the angle $\frac{11\pi}{6}$. This is important since we need the range of θ to actually enclose the regions as we increase from the lower limit to the upper limit. If we'd chosen to use $\frac{11\pi}{6}$ then as we increase from $\frac{7\pi}{6}$ to $\frac{11\pi}{6}$ we would be tracing out the lower portion of the circle and that is not the region that we are after.

So, here are the ranges that will define the region.

$$-\frac{\pi}{6} \leq \theta \leq \frac{7\pi}{6}$$
$$2 \leq r \leq 3 + 2\sin\theta$$

To get the ranges for r the function that is closest to the origin is the lower bound and the function that is farthest from the origin is the upper bound.

The area of the region D is then,

$$\begin{aligned} A &= \iint\limits_D dA \\ &= \int_{-\pi/6}^{7\pi/6} \int_2^{3+2\sin\theta} r\, dr d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \frac{1}{2} r^2 \Big|_2^{3+2\sin\theta} d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \frac{5}{2} + 6\sin\theta + 2\sin^2\theta\, d\theta \\ &= \int_{-\pi/6}^{7\pi/6} \frac{7}{2} + 6\sin\theta - \cos(2\theta)\, d\theta \\ &= \left(\frac{7}{2}\theta - 6\cos\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_{-\frac{\pi}{6}}^{\frac{7\pi}{6}} \\ &= \frac{11\sqrt{3}}{2} + \frac{14\pi}{3} = 24.187 \end{aligned}$$