Partial Derivatives

Alternative Symbols:

If $z = f(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y.$$

Computations of Partial Derivatives

$z = f(x,y)$

$$\frac{\partial z}{\partial x} = Differentiate\ for\ (x), hold\ (y)\ as\ constant$$

$$\frac{\partial z}{\partial y} = Differentiate\ for\ (y), hold\ (x)\ as\ constant$$

Example: Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if:

$$f(x,y) = x^2 + 3xy + y - 1$$

Solution:

To find $\frac{\partial f}{\partial x}$, we treat y as a constant and differentiate with respect to x :

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3y + 0 - 0 = \boxed{2x + 3y}$$

$\therefore$ The value of $\frac{\partial f}{\partial x}$ at $(4,-5)$ is:

$$= (2)(4) + 3(-5) = 8 - 15 = -7$$

To find $\frac{\partial f}{\partial y}$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1) = 0 + 3x + 1 - 0 = \boxed{3x + 1}$$

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$\therefore$ The value of $\dfrac{\partial f}{\partial y}$ at $(4,-5)$ is:

$$=(3)(4)+1=12+1=13$$

Example: Find f_x and f_y if $f(x,y)=\dfrac{2y}{y+\cos x}$

Solution:

We treat f as a quotient with y held constant, we get:

$$f_x=\frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\frac{\partial}{\partial x}(2y)-2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$=\frac{(y+\cos x)(0)-2y(-\sin x)}{(y+\cos x)^2}=\frac{2y\sin x}{(y+\cos x)^2}$$

with x held constant, we get:

$$f_y=\frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right)=\frac{(y+\cos x)\frac{\partial}{\partial y}(2y)-2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}$$

$$=\frac{(y+\cos x)(2)-2y(1)}{(y+\cos x)^2}=\frac{2y+2\cos x-2y}{(y+\cos x)^2}$$

$$=\frac{2\cos x}{(y+\cos x)^2}$$

Example: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x,y)=(2x-3y)^3$

Solution:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(2x-3y)^3=3(2x-3y)^2(2)=\boxed{6(2x-3y)^2}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(2x-3y)^3=3(2x-3y)^2(-3)=\boxed{-9(2x-3y)^2}$$

H.W: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the functions:

1. $f(x,y)=(xy-1)^2$

2. $f(x,y)=x^2-xy+y^2$

3. $f(x,y)=(x^2-1)(y+2)$

4. $f(x,y)=\frac{x}{(x^2+y^2)}$

➢ Functions of three or more variables:

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

$$w=F(x,y,z)$$

$$\frac{\partial w}{\partial x}=Differentiate\ for\ (x), holding\ both\ (y,z) as\ constants$$

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Example: Find $\frac{\partial f}{\partial z}$ if $f(x,y,z) = x\sin(y+3z)$

Solution:

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[x\sin(y+3z)\right] = x\frac{\partial}{\partial z}\sin(y+3z)$$

$$= x\cos(y+3z)\frac{\partial}{\partial z}(y+3z)$$

$$= x\cos(y+3z)(3) = 3x\cos(y+3z)$$

Example: Find f_x, f_y and f_z if $f(x,y,z) = xy + yz + xz$

Solution:

$$f_x = \frac{\partial}{\partial x}(xy + yz + xz) = y + z$$

$$f_y = \frac{\partial}{\partial y}(xy + yz + xz) = x + z$$

$$f_z = \frac{\partial}{\partial z}(xy + yz + xz) = y + x$$

H.W: Find f_x, f_y and f_z if $f(x,y,z) = x - \sqrt{y^2 + z^2}$

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Second – order partial derivatives:

When we differential a function $f(x, y)$ twice, we produce its second – order derivative. These derivatives are usually denoted by:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

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Example: If $f(x,y)=x^2y^3+x^4y$, find $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$

Solution:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2y^3+x^4y)=\boxed{2xy^3+4x^3y}$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2y^3+x^4y)=\boxed{3x^2y^2+x^4}$$

$\therefore$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}(2xy^3+4x^3y)=2y^3+12x^2y$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}(3x^2y^2+x^4)=6x^2y$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}(3x^2y^2+x^4)=6xy^2+4x^3$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}(2xy^3+4x^3y)=6xy^2+4x^3$$

Example:

Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution:

$$f_y = -4xyz + x^2$$
$$f_{yx} = -4yz + 2x$$
$$f_{yxy} = -4z$$
$$f_{yxyz} = -4.$$

Example: Find f_{xx} , f_{yy} , f_{yx} and f_{xy} if $f(x,y) = x\cos y + ye^x$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) = \boxed{\cos y + ye^x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x) = \boxed{-x\sin y + e^x}$$

$\therefore$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos y + ye^x) = ye^x$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-x\sin y + e^x) = -x\cos y$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-x\sin y + e^x) = -\sin y + e^x$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\cos y + ye^x) = -\sin y + e^x$$

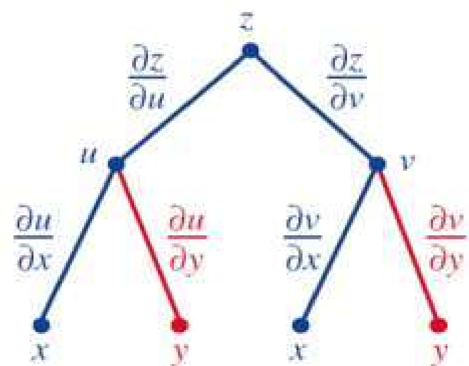
H.W: Find f_{xx} , f_{yy} , f_{yx} and f_{xy} if $f(x,y) = x + y + xy$

The chain Rule for functions of two variables:

$$z = f(u, v)$$

$$u = g(x, y) \text{ and } v = h(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$



Example:

If $z = u^2 - v^3$ and $u = e^{2x-3y}$, $v = \sin(x^2 - y^2)$, find $\partial z/\partial x$ and $\partial z/\partial y$.

Solution:

$$\partial z/\partial u = 2u$$

$$\partial z/\partial v = -3v^2,$$

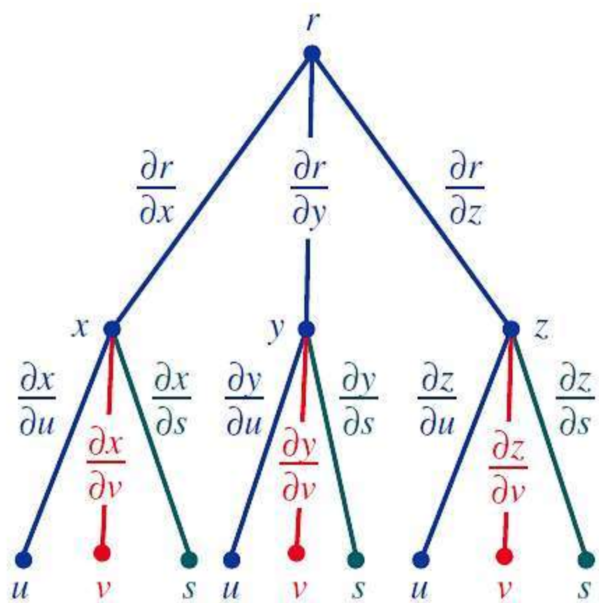
$$\frac{\partial z}{\partial x} = 2u(2e^{2x-3y}) - 3v^2[2x\cos(x^2 - y^2)] = 4ue^{2x-3y} - 6xv^2\cos(x^2 - y^2)$$

$$\frac{\partial z}{\partial y} = 2u(-3e^{2x-3y}) - 3v^2[(-2y)\cos(x^2 - y^2)] = -6ue^{2x-3y} + 6yv^2\cos(x^2 - y^2).$$

Example:

If $r = x^2 + y^5z^3$ and $x = uve^{2s}$, $y = u^2 - v^2s$, $z = \sin(uvs^2)$, find $\partial r/\partial s$.

Solution:



$$\frac{\partial r}{\partial s} = \frac{\partial r}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial r}{\partial z}\frac{\partial z}{\partial s}$$

$$= 2x(2uve^{2s}) + 5y^4z^3(-v^2) + 3y^5z^2(2uvs\cos(uvs^2)).$$

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Example: Use the chain rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. what is the derivatives value at $t = \frac{\pi}{2}$

Solution: we apply the chain rule to find $\frac{dw}{dt}$ as follows:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial (xy)}{\partial x}.\frac{d}{dt}(\cos t) + \frac{\partial (xy)}{\partial y}.\frac{d}{dt}(\sin t)$$

$$= (y)(-\sin t) + (x)(\cos t)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t)$$

$$= -\sin^2 t + \cos^2 t$$

$$= \cos 2t$$

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Example: Use the chain rule to find the derivative of $w = x^2 + y^2$ with respect to t $\left(\dfrac{dw}{dt}\right)$, with $x = \cos t$, $y = \sin t$

Solution: we apply the chain rule to find $\dfrac{dw}{dt}$ as follows:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial}{\partial x}(x^2 + y^2).\frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(x^2 + y^2).\frac{d}{dt}(\sin t)$$

$$= (2x)(-\sin t) + (2y)(\cos t)$$

$$= (2\cos t)(-\sin t) + (2\sin t)(\cos t)$$

$$= -2\cos t \sin t + 2\sin t \cos t$$

$$= 0$$

For check:

$$w = x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

$$\therefore \quad \frac{dw}{dt} = 0$$

H.W: Use the chain rule to find the derivative of $w = x^2 + y^2$ with respect to t $\left(\dfrac{dw}{dt}\right)$, with $x = \cos t + \sin t$, $y = \cos t - \sin t$

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