



## Ordinary Differential Equation

### **Definition of a Differential Equation**

A differential equation is an equation involving derivatives or differentials. The following are some examples of differential equations.

**Ex. 1.**  $(y'')^2 + 3x = 2(y')^3$  where  $y' = dy/dx$ ,  $y'' = d^2y/dx^2$

**Ex. 2.**  $\frac{dy}{dx} + \frac{y}{x} = y^2$

**Ex. 3.**  $\frac{d^2Q}{dt^2} - 3 \frac{dQ}{dt} + 2Q = 4 \sin(2t)$

**Ex. 4.**  $\frac{dy}{dx} = \frac{x+y}{x-y}$  or  $(x+y) dx + (y-X) dy = 0$

**Ex. 5.**  $\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0$

Equations such as those in examples (1 – 4) involving only one independent variable are called **ordinary differential equations**. Equation such as that of example 5 or more independent variables are called **partial differential equations**.

### **Order and degree of a differential equation**

**Order:** (highest derivative), **Degree:** (Power of highest derivative)

**Ex. 6.**

1-  $x^2 \bar{y} + \bar{y} + (x^2 - 4)y = 0$  Ordinary, Order 2, Degree 1

2-  $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^5 + \frac{y}{x^2 + 1} = e^x$  Ordinary, Order 3, Degree 2

3-  $\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$  Partial, Order 4, Degree 1

4-  $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$  Partial, Order 2, Degree 1



### **3- Solution of first order ordinary differential equations (O.D.E.)**

#### **1- First order: variable separable**

A first order differential equation can be solved by integration if it is possible to collect all y-terms with dy and all x-terms with dx, that is, if it is possible to write the equation in the form:

$$M(x) dx + N(y) dy = 0$$

than the general solution:

$$\int M(x) dx + \int N(y) dy = c$$

Where:

c is an arbitrary constant

**Ex. 1. :** Solve the differential equation:-

$$\frac{dy}{dx} = (1 + y^2)e^x$$

**Solution**

$$e^x dx - \frac{1}{1 + y^2} dy = 0$$

$$\int e^x dx - \int \frac{dy}{1 + y^2} = \int 0$$

$$e^x - \tan^{-1} y = c$$

$$\tan^{-1} y = e^x - c$$



$$y = \tan(e^x - c)$$

**Ex. 2. :** Solve the differential equation:-

$$\frac{dy}{dx} = \frac{1}{x}y = 0$$

**Solution**

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

$$\frac{dy}{y} + \frac{dx}{x} = 0$$

$$\int \frac{dy}{y} + \int \frac{dx}{x} = c$$

$$\ln y + \ln x = c$$

$$\ln xy = c$$

$$yx = e^c$$

$$y = \frac{e^c}{x}$$



**Ex. 3. :** Solve the differential equation:-

$$\frac{dy}{dx} = e^{x-y}$$

**Solution**

$$\frac{dy}{dx} = \frac{e^x}{e^y}$$

$$\int e^y dy = \int e^x dx$$

$$e^y = e^x + c$$

$$\ln e^y = \ln(e^x + c)$$

$$y = \ln(e^x + c)$$

**H.W.:** Solve the differential equation:-

1)  $x \frac{dy}{dx} + 3y = 2$       Given       $y = 0$  ,  $x = 0$

2)  $x(2y - 3) dx + (x^2 + 1) dy = 0$

3)  $x^2(y^2 + 1) dx + y\sqrt{x^3 + 1} dy = 0$

4)  $x^2 (y + 1) dx + y^2 (x - 1) dy = 0$



## 2 - First order: homogenous

An equation of the form:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \dots (1)$$

Is called a homogenous equation. In this equation, the variable cannot be separated but can be transformed by a change of variable into an equation where the variables can be separated.

To transform a homogenous equation such equation (1) into an equation whose variable may separate,

We introduce the new independent variable:

$$v = \frac{y}{x}, \quad \frac{1}{v} = \frac{x}{y} \dots \dots \dots (2)$$

$$y = vx, \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$$

substitute in (1)  $v + x \frac{dv}{dx} = F(v)$

this equation is a variable separable  $\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$

Then after solving this equation, the original eq. maybe obtained by substituting  $\frac{y}{x}$  for  $v$ .

**Ex. 1.:** Show that the equation is homogenous and find solutions satisfied condition  $y(1) = 1$ :-

$$\frac{dy}{dx} = \frac{-(x^2 + y^2)}{2xy}$$

**Solution**



$$\frac{dy}{dx} = - \left( \frac{\frac{x^2}{x^2} + \frac{y^2}{x^2}}{\frac{2xy}{x^2}} \right) = - \left( \frac{1 + \left(\frac{y}{x}\right)^2}{\frac{2y}{x}} \right)$$

$$\frac{dy}{dx} = - \left( \frac{1 + v^2}{2v} \right) = F(v)$$

$$\frac{dy}{x} + \frac{dv}{v - \left( - \left( \frac{1+v^2}{2v} \right) \right)} = 0$$

$$\frac{dy}{x} + \frac{dv}{v + \left( \frac{1+v^2}{2v} \right)} = 0$$

$$\frac{dy}{x} + \frac{dv}{\frac{2v^2 + 1 + v^2}{2v}} = 0$$

$$\frac{dy}{x} + \frac{2v dv}{1 + 3v^2} = 0$$

$$\int x^{-1} dx + \frac{1}{3} \int (1 + 3v^2)^{-1} \cdot 6v dv = \int 0 dx$$

$$\ln x + \frac{1}{3} \ln(1 + 3v^2) = c$$

$$3 \ln x + \ln(1 + 3v^2) = 3c$$



$$\ln x^3 (1 + 3v^2) = 3c$$

$$x^3 \left(1 + 3 \frac{y^2}{x^2}\right) = e^{3c} = c' \quad \text{Let } c' = e^{3c}$$

$$x^3 + 3xy^2 = c'$$

$$(1)^3 + 3(1)(1)^2 = c'$$

$$c' = 1 + 3 = 4$$

$$x^3 + 3xy^2 = 4$$

**Ex. 2.:** Solve the differential equation:-

$$(x^2 + y^2)dx + xydy = 0$$

**Solution**

$$xydy = -(x^2 + y^2)dx$$

$$\frac{dy}{dx} = \frac{-(x^2 + y^2)}{xy} = \frac{-\left(\frac{x^2}{x^2} + \frac{y^2}{x^2}\right)}{\frac{xy}{x^2}} = \frac{-\left(1 + \left(\frac{y}{x}\right)^2\right)}{\frac{y}{x}}$$

$$\frac{dy}{dx} = \frac{-(1 + v^2)}{v} = F(v)$$

$$\frac{dx}{x} + \frac{dv}{v - F(v)} = 0$$



$$\frac{dx}{x} + \frac{dv}{v - \left( \frac{-(1+v^2)}{v} \right)} = 0$$

$$\frac{dx}{x} + \frac{dv}{v + \frac{1+v^2}{v}} = 0$$

$$\frac{dx}{x} \frac{dv}{\frac{v^2 + 1 + v^2}{v}} = 0$$

$$\frac{dx}{x} + \frac{v dv}{2v^2 + 1} = 0$$

$$\int x^{-1} dx + \frac{1}{2} \int (2v^2 + 1)^{-1} \cdot 2v dv = \int 0 dx$$

$$\ln x + \frac{1}{2} \ln(2v^2 + 1) = c$$

$$2 \ln x + \ln(2v^2 + 1) = 2c$$

$$\ln(x^2)(2v^2 + 1) = 2c$$

$$\ln(x^2) \left( \frac{2y^2}{x^2} + 1 \right) = 2c$$

$$\ln(2y^2 + x^2) = 2c$$

$$2y^2 + x^2 = e^{2c}$$

$$2y^2 + x^2 = c'$$



**H.W.: Solve the differential equation:-**

1)  $(x^2 + y^2) dx + xy dy = 0$

2)  $x^2 dy + (y^2 - xy) dx = 0$

3)  $x e^{y/x} dx + y dx - x dy = 0$