Heat Conduction Equation

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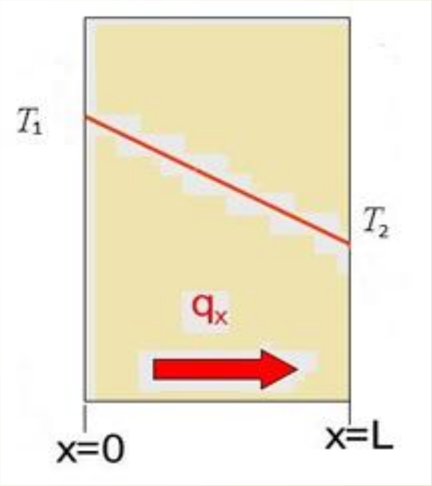
Heat Conduction Equation

- One dimensional (Plane wall)
- Steady state with no heat generation is $\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) = 0$
- When A=constant, and
- K= constant, then
- $\frac{d^2T}{dx^2} = 0$

Heat Conduction Equation

- To integrate this equation we will fixed the boundary conditions at first
- At X=0 $T = T_1$
- At X=L $T = T_2$
- $\frac{d^2T}{dx^2} = 0 \rightarrow \frac{d}{dx}\left(\frac{dT}{dx}\right) = 0$
- By first integration
- $\left(\frac{dT}{dx}\right) = C_1$
- where C_1: integration constant

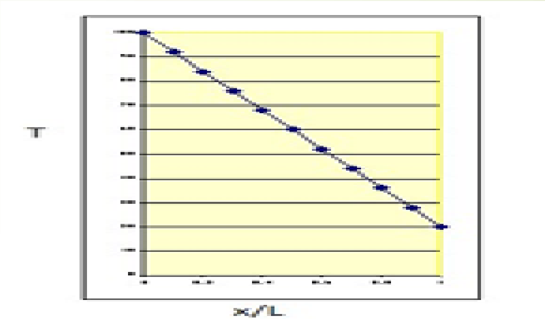


Heat Conduction Equation

- And by second integration
- $T = C_1 x + C_2$
- Applying the Boundary conditions
- B.C.1 $T_1 = C_1 \times 0 + C_2 \rightarrow C_2 = T_1$
- B.C.2 $T_2 = C_1 \mathrm{L} + T_1 \rightarrow C_2 = \frac{1}{L}(T_2 - T_1)$
- Then the temperature distribution equation (T.D.E.) is
- $\mathrm{T}(x) = (T_2 - T_1)\frac{x}{L} + T_1$

Heat Conduction Equation

Or $T(x) = \left(1 - \frac{x}{L}\right) T_1 + \frac{x}{L} T_2$



- **Example 1. Wall of thickness 0.5m. The temperature at one Side is $100^o C$ and at other side is 20^o. Find the temperature distribution equation, and the heat transfer per unit area. Thermal conductivity is 2W/$m.^o\ C$**

Example

- Solution: Plane Wall with boundary conditions
- B.C.1 x=0 $T = T_1 = 100^o C$
- B.C.2 x=L=0.5m $\mathrm{T} = T_2 = 20^o C$
- We can repeat the solution
- $\frac{d^2x}{dx^2} = 0$ by double integration, we get
- $T = (T_2 - T_1)\frac{x}{L} + T_1 = \left(1 - \frac{x}{L}\right) T_1 + \frac{x}{L} T_2$
- $T = \left(1 - \frac{x}{0.5}\right) 100 + \frac{x}{0.5} 20$

Example

- $q = k\frac{T_1 - T_2}{\Delta x} = 2\frac{100-20}{0.5} = 320W$
- Find the temperature at mid point of the wall
- At mid point x=0.25m
- Then $T = \left(1 - \frac{x}{0.5}\right)100 + \frac{x}{0.5}20$
- $T = \left(1 - \frac{0.25}{0.5}\right)100 + \frac{0.25}{0.5}20 = 60^o C$

Heat Conduction Equation

- Plane Wall with thermal conductivity function of temperature
- $k = k_o + k_1 T$
- Where k_o *and* k_1 are constant
- $\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \rightarrow \frac{d}{dx}\left((k_o + k_1 T)\frac{dT}{dx}\right) = 0$
- By integrating $(k_o + k_1 T)\frac{dT}{dx} = C_1$
- $(k_o + k_1 T)dT = C_1 dx$
- Second integrating $\left(k_o T + \frac{1}{2}k_1 T^2\right) = C_1 x + C_2$

- By applying the Boundary conditions that are

X=0 $T = T_1$ and x=L $T = T_2$

$$\left(k_o T_1 + \frac{1}{2} k_1 T_1^2\right) = C_1 \times (0) + C_2 = C_2$$

And $\left(k_o T_2 + \frac{1}{2} k_1 T_2^2\right) = C_1 L + C_2$

Then $C_2 = \left(k_o T_1 + \frac{1}{2} k_1 T_1^2\right)$ And

$$C_1 = \frac{\left(k_o (T_2 - T_1) + \frac{1}{2} k_1 (T_2^2 - T_1^2)\right)}{L}$$

Finally the T.D.E

- By rearranging the T.D.E

$$k_1T^2 + 2k_oT - 2(C_1x + C_2)=0$$

- By using Rule equation we get that

- $T = \frac{-2k_o \pm \sqrt{(2k_o)^2 + 8k_1(C_1x + C_2)}}{2k_1}$

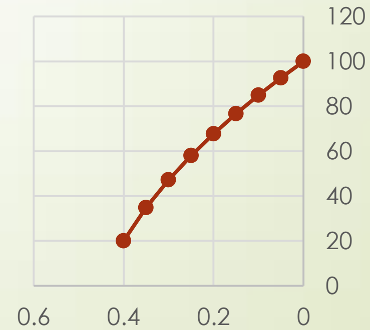
- **Example 2. wall of thickness 0.4m its temperatures of the two side of the wall are 100^oC and 20^oC. Thermal conductivity of wall material is (2+0.05T). Find the T.D.E. and heat transfer through the wall.**

- Solution: heat conduction through the wall

- Given data: wall thickness Δx=L=0.4m and thermal conductivity $\mathrm{k} = k_o + k_1 T$
- $k = 2 + 0.05T$. The boundary conditions are
- At x=0m T=$T_1 = 100^o C$,
- At x=0.4m $T = T_2 = 20^o C$
- The differential equation is
- $\frac{d}{dx}\left((k_o + k_1 T)\frac{dT}{dx}\right) = \frac{d}{dx}\left((2 + 0.05T)\frac{dT}{dx}\right) = 0$

- By integration we get $(2 + 0.05T)\frac{dT}{dx} = C_1$
- $(2 + 0.05T)dT = C_1 dx$
- And second integration
- $2T + \frac{0.05}{2}T^2 = C_1 x + C_2$
- $C_2 = \left(2 \times 100 + \frac{0.05}{2}(100)^2\right) = 450$
- $C_1 = \frac{\left(2(20-100)+\frac{1}{2}0.05(20^2-100^2)\right)}{0.4} = 1000$
- $T = \frac{-2k_o \pm \sqrt{(2k_o)^2+8k_1(C_1 x+C_2)}}{2k_1}$
- $T = \frac{-2\times2 \pm \sqrt{(2\times2)^2+8(0.05)(-1000x+450)}}{2(0.05)}$
- $T = \frac{-4+\sqrt{16+(180-400x)}}{0.1}$
- Now to find the heat transfer by conduction
- Through this wall

X	T
0	100
0.05	92.66
0.1	84.90
0.15	76.62
0.2	67.70
0.25	57.98
0.3	47.18
0.35	34.83
0.4	20



- Q=-Ak$\frac{dT}{dx} \rightarrow q = -(k_o + k_1 T)\frac{dT}{dx}$
- By separation of variables
- $\int_1^2 q dx = -\int_1^2 (k_o + k_1) dT$
- $q(x_2 - x_1) = -\left(k_o T - k_1 \frac{T^2}{2}\right)_1^2$
- $q = -\frac{k_o(T_2 - T_1) + \frac{k_1}{2}(T_2^2 - T_1^2)}{\Delta x}$
- $q = -\left(k_o + \frac{k_1}{2}(T_1 + T_2)\right)\frac{T_2 - T_1}{L}$

$$= \left(2 + \frac{0.05}{2}(100 + 20)\right)\frac{100-20}{0.4} = 5\,\frac{100-20}{0.4} = 1000W/m^2$$

Plane wall conduction with heat generation

- The wall with heat generation or heat source at steady state, its differential equation of temperature distribution is:
- $\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\ddot{g}=0$ where $\ddot{g}$ is heat generation per unit volume (some times is denoted $\ddot{q}$
- Where the thermal conductivity is constant
- k= constant, then
- $\frac{d^2T}{dx^2}=-\frac{\ddot{q}}{k}$

solution this equation is by integration, and that required two boundary condition depending on the problem.

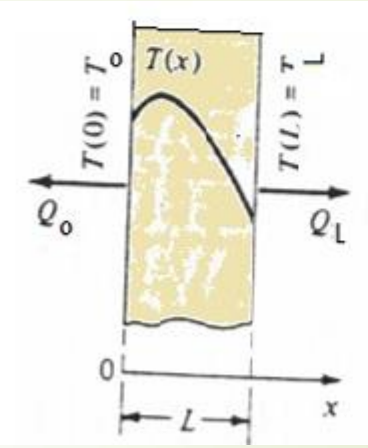
1- wall is of constant thermal conductivity k and heat generation(source) $\ddot{g}$, its thickness L and one surface is at T_o and the other surface at T_L. It is wonted to find the temperature distribution through this wall.

The differential equation used is $\frac{d^2T}{dx^2} = -\frac{\ddot{g}}{k}$

Boundary conditions are

At x=0 $T = T_o$ and at x=L $T = T_o$

$$\frac{d^2x}{dx^2} = -\frac{\ddot{g}}{k}$$



- By first integrating
- $\frac{dT}{dx} = -\frac{\ddot{g}x}{k} + C_1$
- Second integration gives
- $\boldsymbol{T = -\frac{\ddot{g}x^2}{2k} + C_1 x + C_2}$ **T.D.E**
- Where C_1 and C_2 are integration constants
- From B.C. 1 where x=0 $T = T_o$
- $T_o = -\frac{\ddot{g}(0)^2}{2k} + C_1(0) + C_2 \rightarrow \; C_2 = T_o$

- From B.C.2 where x=L $T = T_L$

- $T_L = -\frac{\ddot{g}L^2}{2k} + C_1 L + T_o$
- Then $C_1 = \frac{1}{L}(T_L - T_o) + \frac{\ddot{g}L}{2k}$
- The T.D.E which is $T = -\frac{\ddot{g}x^2}{2k} + C_1 x + C_2$
- $T = -\frac{\ddot{g}x^2}{2k} + \left[\frac{1}{L}(T_L - T_o) + \frac{\ddot{g}L}{2k}\right] x + T_o$
- $\boldsymbol{T = \frac{\ddot{g}x}{2k}(L - x) + (T_L - T_o)\frac{x}{L} + T_o}$
- To prove that this equation is write substituting the x values at boundary locations, it gives the same values of temperatures as in B.Cs

- To find the heat flux at each surface we will use the following: $q = -k\frac{dT}{dx}$
- $\dot{q}_0 = \left(-k\frac{dT}{dx}\right)_{x=0} = -k\frac{d}{dx}\left(\frac{\ddot{g}x}{2k}(L-x) + (T_L - T_o)\frac{x}{L} + T_o\right)$=-k$\left(\frac{\dot{q}}{2k}(L - 2x) + (T_L - T_o)\frac{1}{L}\right)_{x=0}$=$-\left(\frac{\ddot{g}L}{2} + \frac{k(T_L - T_o)}{L}\right)$
- $\dot{q}_L = \left(-k\frac{dT}{dx}\right)_{x=L} = -k\left(\frac{-\ddot{g}L}{2k} + \frac{T_L - T_o}{L}\right)_L = \frac{\ddot{g}L}{2} - \frac{k(T_L - T_o)}{L}$
- We find the total heat transfer is equal to heat generation
- $g\ddot{L} = |\dot{q}_0| + |\dot{q}_L| = \left(\frac{\ddot{g}L}{2} + \frac{k(T_L - T_o)}{L}\right) + \left(\frac{\ddot{g}L}{2} - \frac{k(T_L - T_o)}{L}\right) = \frac{\ddot{g}L}{2} + \frac{\ddot{g}L}{2} = \ddot{g}L$

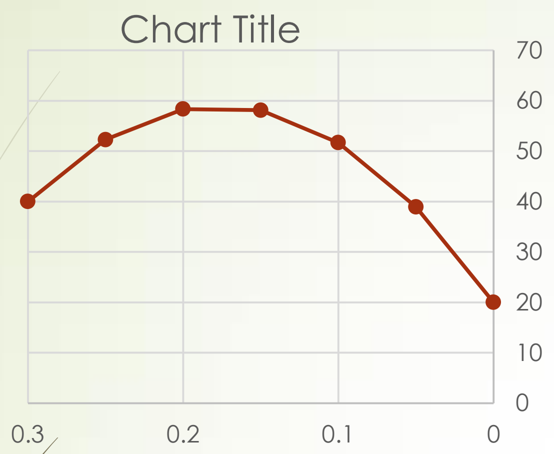
- To find the location and magnitude of the maximum temperature in the wall, we will derive the T.D.E by x and equal it to zero:
- $\boldsymbol{T = \frac{\ddot{g}}{2k}\left(Lx - x^2\right) + (T_L - T_o)\frac{x}{L} + T_o}$
- $\frac{dT}{dx} = \frac{\ddot{g}}{2k}(L - 2x) + (T_L - T_o)\frac{1}{L} = 0$
- $2x - L = \frac{2k}{\ddot{g}L}(T_L - T_o)$
- $x = \frac{L}{2} + \frac{k}{\ddot{g}L}(T_L - T_o)$

By substituting this T.D.E we will find the maximum temperature.

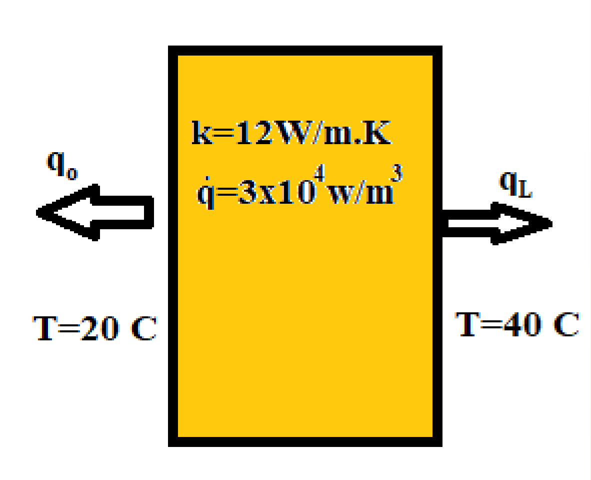
- Example: **A plane wall of thickness 30cm and its material is of thermal conductivity 12W/m.^{o}C the heat generation in the wall is $3 \times 10^4 W/m^3$. The wall surfaces temperatures are $20^o C$ and $40^o C$. Find the temperature distribution equation (T.D.E), location of maximum temperature and its value. The heat transfer from each surface per unit area.**
- Solution: pane wall $L = 30cm,$ k=12W/m .^{o}C, $\ddot{g} = 3 \times 10^4 W/m^3$ the temperatures are $T_o = 20^o C$, $T_L = 40^o C$.
- Properties: The properties are constant.

Assumption: the condition is steady one-dimension with heat generation.

- Analysis: The differential equation for plane wall with heat generation and steady state is
- $\frac{d^2T}{dx^2} = -\frac{\ddot{g}}{k}$ and
- B.C.1 x=0 $T = T_o,\ B.C.2\ \ x = L\ \ T = T_l$
- $\boldsymbol{T = \frac{\ddot{g}}{2k}\left(Lx - x^2\right) + (T_L - T_o)\frac{x}{L} + T_o}$
- $\boldsymbol{T = \frac{3\times10^4}{2\times12}\left(0.3x - x^2\right) + (40 - 20)\frac{x}{0.3} + 20}$
- $\boldsymbol{T = 1250\left(0.3x - x^2\right) + \frac{20x}{0.3} + 20}$
- **This is the T.D.E**



X m	T C
0	20
0.05	38.958
0.10	51.667
0.15	58.125
0.1767	59.028
0.20	58.333
0.25	52.291
0.3	40



- To the location of the maximum temperature

$$T = 1250(0.3x - x^2) + \frac{20x}{0.3} + 20$$

- $\frac{dT}{dx} = 1250(0.3 - 2x) + \frac{20}{0.3} = 0$
- $2x = 0.3 + \frac{20}{0.3\times1250} = 0.3533$
- $x = \frac{0.3533}{2}$=0.1767m=17.67cm
- $T_{max} = 1250(0.3 \times 0.1767 - (0.1767)^2) + \frac{20(0.1767)}{0.3} + 20$ $= 59.028^o C$
- Now to find heat flux at x=0m and x=0.3m

- $q_o = \left(-k\frac{dT}{dx}\right)_{x=0} = -12\left[\frac{dT}{dx} = 1250(0.3 - 2 \times 0) + \frac{20}{0.3}\right]$

  $= -5300\text{W}/m^2$

- $q_L = \left(-k\frac{dT}{dx}\right)_{x=L} = -12\left[\frac{dT}{dx} = 1250(0.3 - 2 \times 0.3) + \frac{20}{0.3}\right]$
  $= 3700W/m^2$
- We can see that heat generation is equal to the heat flow from the two surfaces
- That $\ddot{g} = \frac{q_L + q_o}{L} = \frac{3700+5300}{0.3} = \frac{30000W}{m^3} = 3 \times \frac{10^4 W}{m^3}$

2- Heat generation in a wall of the same surface temperature

- Let us take a wall of thickness 2L and thermal conductivity k. The heat generation in the wall is $\ddot{g} W/m^3$. The temperatures of the two sides are T_w. $Find\ T.D.E$
- The Differential equation is $\frac{d^2T}{dx^2} = -\frac{\ddot{g}}{k}$
- The boundary conditions are
- $x = \pm L \qquad T = T_w, \qquad x = 0 \qquad \frac{dT}{dx} = 0$
- $T_{max} = T_o \qquad at\ \ x = 0$

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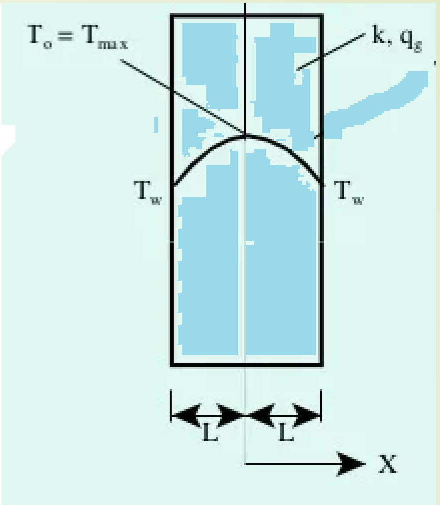
$$\frac{d^2T}{dx^2} = -\frac{\ddot{g}}{k}$$ by integration



- $\frac{dT}{dx} = -\frac{\ddot{g}x}{k} + C_1$
- From B.C.2 where $\frac{dT}{dx} = 0\ at\ x = 0$
- $0 = 0 + C_1 \rightarrow C_1 = 0$
- The differential equation becomes
- $\frac{dT}{dx} = -\frac{\ddot{g}x}{k}$ and by integrating this equation we get
- $T = -\frac{\ddot{g}x^2}{2k} + C_2$ B.C.1 x=$\pm L$ $T = T_w$

- $T_w = -\frac{\ddot{g}L^2}{2k} + C_2 \rightarrow \quad C_2 = \frac{\ddot{g}L^2}{2k} + T_w$
- The T.D.E is $\quad T = -\frac{\ddot{g}x^2}{2k} + \frac{\ddot{g}L^2}{2k} + T_w$
- OR $\quad T - T_w = \frac{\ddot{g}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right) \quad$ (1)
- At the mid-line where x=0, $T = T_o$
- $T_o - T_w = \frac{\ddot{g}L^2}{2k} \quad$ (2)
- By dividing eq.(1) by eq.(2), we get that
- $\frac{T - T_w}{T_o - T_w} = \left(1 - \frac{x^2}{L^2}\right) \quad$ (3)

Conduction Equation for Cylindrical wall

- Heat conduction equation for cylindrical wall with no heat generation becomes
- $\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$ -------------------(1)
- And for k=constant, it becomes
- $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$ -------------------(2)
- If we take long cylinder with inner radius of r_i with temperature at that surface is T_i, and outer radius of r_o with temperature T_o. To find the T.D.E through this cylindrical wall, we take the eq.(2)

- $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$ and the boundary conditions are:
  B.C.1 at $r = r_i$ $T = T_i$
- B.C.2 at $r = r_o$ $T = T_o$
- By integrating the differential equation we get
- $r\frac{dT}{dr} = C_1 \quad \rightarrow \quad \frac{dT}{dr} = \frac{C_1}{r}$
- By second integrating it becomes
- $T = C_1 lnr + C_2$
- By substituting the B.Cs, we get
- $T_i = C_1 lnr_i + C_2$ and $T_o = C_1 lnr_o + C_2$

- By subtracting the first equation from the second, we get that:
- $T_o - T_i = C_1 ln\frac{r_o}{r_i} \rightarrow C_1 = \frac{T_o - T_i}{ln\frac{r_o}{r_i}}$
- By substituting in one of the upper relation we get
- $T_i = \frac{T_o - T_i}{ln\frac{r_o}{r_i}} lnr_i + C_2$
- $C_2 = T_i - \frac{T_o - T_i}{ln\frac{r_o}{r_i}} lnr_i$
- By substituting the values of C_1 *and* C_2 in the T.D.E
- $T = C_1 lnr + C_2$

- $T = C_1 lnr + C_2$
- $T = \frac{T_o - T_i}{ln\frac{r_o}{r_i}} lnr + T_i - \frac{T_o - T_i}{ln\frac{r_o}{r_i}} lnr_i$
- $\boldsymbol{T - T_i = \frac{T_o - T_i}{ln\frac{r_o}{r_i}} (lnr - lnr_i) = \frac{T_o - T_i}{ln\frac{r_o}{r_i}} \ln\frac{r}{r_i}}$
- $\boldsymbol{\frac{T - T_i}{T_o - T_i} = \frac{\ln\frac{r}{r_i}}{ln\frac{r_o}{r_i}}}$ T.D.E through cylindrical wall with no heat generation.

- **Example:** **Find the temperature at the mid-point of pipe with inner diameter of 20cm and outer diameter of 50cm. The temperature at inner surface is 125^oC and at outer surface is 25^oC. If the thermal conductivity of pipe material is 10W/$m.^o\,C$, Find the heat transfer from the tube surface per 10m long.**
- Solution: Pipe with no heat generation. Its specifications are
- $D_i = 20cm = 0.2m, r_i = 0.1m\ \ \&\ \ T_i = 125^oC,$
- $D_o = 50cm = 0.5m\ \ ,\ \ r_o = 0.25m\ \ \ \&\ \ T_o = 25^oC,\ k = 10W/m.^o\,C$ pipe length ι=10m

- It is needed to find the temperature at the mid-point of the wall, and the heat transfer from the pipe if its length is 10m.
- At the mid-point the radius become
- $r_m = \frac{r_i + r_o}{2} = \frac{0.1+0.25}{2} = 0.175m$
- Or $r_m = r_i + \frac{\Delta r}{2} = 0.1 + \frac{0.25-0.1}{2} = 0.1 + 0.075 = 0.175m$
- By using T.D.E for cylindrical wall
- $\frac{T-T_i}{T_o-T_i} = \frac{\ln\frac{r}{r_i}}{ln\frac{r_o}{r_i}} \rightarrow \frac{T-125}{25-125} = \frac{ln\frac{0.175}{0.10}}{ln\frac{0.25}{0.10}} = 0.61$
- $T = 125 - (100 \times 0.61) = 63.9^o C$
- $Q = \frac{T_i - T_o}{\frac{1}{2\pi Lk} ln\frac{r_o}{r_i}} = \frac{125-25}{\frac{1}{2\pi\times10\times10} ln\frac{0.25}{0.1}} = 68572W = 68.572kW$

Conduction In cylindrical wall with heat generation

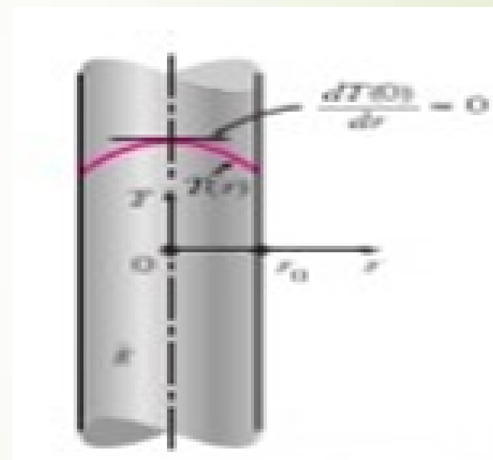
The differential equation of temperature distribution through cylindrical wall with heat generation is

$$\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right)+\ddot{g}=0$$

For k=constant. Equation becomes

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=-\frac{\ddot{g}r}{k}$$

- **Solid cylindrical pipe** of radius R with heat generation per unit volume $\ddot{g} W/m^3$ and constant thermal conductivity k $W/m.^o C$
- Boundary conditions are
- B.C.1 $\frac{dT}{dr} = 0$ at r=0
- B.C.2 $T = T_R$ at r=R
- The integrating $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\ddot{g}r}{k}$
- $r\frac{dT}{dr} = -\frac{\ddot{g}r^2}{2k} + C_1 \rightarrow \frac{dT}{dr} = -\frac{\ddot{g}r}{2k} + \frac{C_1}{r}$
- By applying B.C.1 $C_1 = 0$
- Then the equation becomes



- $\frac{dT}{dr} = -\frac{\ddot{g}r}{2k}$
- By the second integration we get that
- $T = -\frac{\ddot{g}r^2}{4k} + C_2$ T.D.E
- from B.C.2 r=R T=T_R
- $T_R = -\frac{\ddot{g}R^2}{4k} + C_2 \rightarrow C_2 = \frac{\ddot{g}R^2}{4k} + T_R$
- $T = -\frac{\ddot{g}r^2}{4k} + C_2 = -\frac{\ddot{g}r^2}{4k} + \frac{\ddot{g}R^2}{4k} + T_R$
- $\boldsymbol{T = \frac{\ddot{g}R^2}{4k}\left(1 - \left(\frac{r}{R}\right)^2\right) + T_R}$

- $T - T_R = \frac{\ddot{g}R^2}{4k}\left(1 - \left(\frac{r}{R}\right)^2\right)$ (1)

- It is T.D.E in cylindrical wall with heat source
- At the center of cylinder where r=0 $T = T_o$
- $T_o - T_R = \frac{\ddot{g}R^2}{4k}\left(1 - \left(\frac{0}{R}\right)^2\right)$
- $T_o - T_R = \frac{\ddot{g}R^2}{4k}$ (2)
- By dividing eq.(1) by eq.(2) we get
- $\frac{T - T_R}{T_o - T_R} = \left[1 - \left(\frac{r}{R}\right)^2\right]$ it is also T.D.E

- To calculate heat transfer from the surface of the cylinder per length.
- $\dot{q}_R = -k\frac{dT}{dr} = -k\left(-\frac{\ddot{g}r}{2k}\right)_{r=R} = \frac{\ddot{g}R}{2}$
- $\dot{Q}_R = 2\pi LR\frac{\ddot{g}R}{2} = \pi LR^2\ddot{g}$

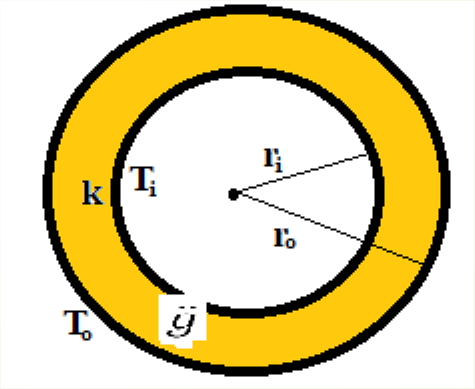
- **Example:** **Solid pipe of diameter 20cm and its outer surface temperature is 25^oC. The thermal conductivity of its material is 20W/$m.^o C$. The heat generation is $10^5 W/m^3$. Find the temperature at the center, and heat transfer from the cylinder outside surface.**
- **Solution:** Solid pipe with heat cylinder.
- D=20cm, R=10cm=0.1m, the heat generation $\ddot{g} = 10^5 W/m^3, T_R = 25^o C$
- **Properties:** Constant thermal conductivity k=20W/$m.^o C$.

- **Assumption:** Steady state heat conduction

- **Analysis:** the temperature at the center of the cylinder.
- $T_o - T_R = \frac{\ddot{g}R^2}{4k}$
- $T_o = \frac{10^5(0.1)^2}{4(20)} + 25 = 37.5^oC$
- The heat transfer from the surface
- $\dot{Q}_R = \pi L R^2 \ddot{g}$
- $\dot{Q}_R = \pi \times 1 \times (0.1)^2 \dot{\times} 10^5 = 1000\pi W$

Heat generation in hollow cylinder

- A hollow cylinder of inner radius r_i and outer radius r_o. The temperature at inner surface is T_i and at outer surface is T_o. Thermal conductivity of the pipe material is k, and the heat generation per volume $\ddot{g}$.
- The differential equation is
- $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\ddot{g}r}{k}$
- The boundary conditions are
- At $r = r_i$ $T = T_i$
- at $r = r_o$ $T = T_o$



- By integrating differential equation first integrating

$$\frac{dT}{dr} = -\frac{\ddot{g}r}{2k} + \frac{C_1}{r}$$

- By second integrating
- $\boldsymbol{T = -\frac{\ddot{g}r^2}{4k} + C_1 lnr + C_2}$ (T.D.E)
- From B.C 1 $T_i = -\frac{\ddot{g}r_i^2}{4k} + C_1 lnr_i + C_2$ (1)
- And B.C 2 $T_o = -\frac{\ddot{g}r_o^2}{4k} + C_1 lnr_o + C_2$ (2)

- By subtracting eq.(2) from eq.(1) we get
- $T_i - T_o = -\frac{\ddot{g}r_i^2}{4k} + \frac{\ddot{g}r_o^2}{4k} + C_1 lnr_i - C_1 lnr_o$
- $T_i - T_o = \frac{\ddot{g}}{4k}\left(r_o^2 - r_i^2\right) - C_1 ln\frac{r_o}{r_i}$
- From this we get that
- $C_1 = \frac{\frac{\ddot{g}}{4k}(r_o^2 - r_i^2) - (T_i - T_o)}{ln\frac{r_o}{r_i}}$
- By substituting value of C_1 in eq.(1) we get:
- $T_i = -\frac{\ddot{g}r_i^2}{4k} + \frac{\frac{\ddot{g}}{4k}(r_o^2 - r_i^2) - (T_i - T_o)}{ln\frac{r_o}{r_i}} lnr_i + C_2$

- $C_2 = \frac{\ddot{g}r_i^2}{4k} - \frac{\frac{\ddot{g}}{4k}(r_o^2 - r_i^2) - (T_i - T_o)}{ln\frac{r_o}{r_i}} lnr_i + T_i$
- After we find C_1 and C_2, substitute them in equation of T.D.E
- $T = -\frac{\ddot{g}r^2}{4k} + C_1 lnr + C_2$
- $T = -\frac{\ddot{g}r^2}{4k} + \frac{\frac{\ddot{g}}{4k}(r_o^2 - r_i^2) - (T_i - T_o)}{ln\frac{r_o}{r_i}} lnr + \frac{\ddot{g}r_i^2}{4k} - \frac{\frac{\ddot{g}}{4k}(r_o^2 - r_i^2) - (T_i - T_o)}{ln\frac{r_o}{r_i}} lnr_i + T_i$

- $$T = -\frac{\ddot{g}}{4k}\left(r^2 - r_i^2\right) + \left[\frac{\ddot{g}}{4k}\left(r_o^2 - r_i^2\right) - (T_i - T_o)\right]\frac{ln\frac{r}{r_i}}{ln\frac{r_o}{r_i}} + T_i$$

  final form of T.D.E
- To find the location and magnitude of maximum temperature:
- $$\frac{dT}{dr} = -\frac{\ddot{g}r}{2k} + \left[\frac{\ddot{g}}{4k}\left(r_o^2 - r_i^2\right) - (T_i - T_o)\right]\frac{r_{i/r}}{r_i ln\frac{r_o}{r_i}} = 0$$
- $$r = \left\{\frac{2k}{\ddot{g}}\left[\frac{\ddot{g}}{4k}\left(r_o^2 - r_i^2\right) - (T_i - T_o)\right]\frac{1}{ln\frac{r_o}{r_i}}\right\}^{1/2}$$
- By substituting in T.D.E we can find the max. temp.

- **Example:** Cylindrical pipe of inner radius 20cm and outer radius 50cm. The temperature of inner surface is 50^oC and of outer surface is 10^oC. Thermal conductivity of the pipe material is 10W$/m.^o\,C$. The heat generation in the pipe wall is $4x10^4 W/m^3$. Find the equation of temperature distribution through the wall of the pipe and find the temperature at the mid-point of the wall. The location and magnitude of the maximum temperature and also the heat flux at inner and outer surface.

- **Solution:** Hollow pipe $r_i = 20cm = 0.2m,$
- $r_o = 50cm \quad = 0.5m,$
- the thermal conductivity k=10W/m.$^o\,C$.
- The heat generation is $\ddot{g} = 4 \times 10^4 W/m^3$.
- B.C.1 At r=$r_i = 0.2m \quad T_i = 50^o C, \quad and$
- B.C.2 at r=$r_o = 0.5m \quad T_o = 10^o C$.
- **Assumption:** The thermal conductivity is constant and heat generation is also constant
- **Analysis:** the differential equation for temperature in a cylindrical wall with heat generation is
- $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\ddot{g}r}{k}$
- By double integration for this equation and substituting the boundary condition:

- $r_i = 0.2m \;\; T_i = 50^oC, \quad r_o = 0.5m \;\; T_o = 10^oC$

$$T = -\frac{\ddot{g}}{4k}\left(r^2 - r_i^2\right) + \left[\frac{\ddot{g}}{4k}\left(r_o^2 - r_i^2\right) - (T_i - T_o)\right]\frac{ln\frac{r}{r_i}}{ln\frac{r_o}{r_i}} + T_i$$

- $T = -\frac{4\times10^4}{4\times10}(r^2 - (0.2)^2)$
  $+ \left[\frac{4\times10^4}{4\times10}(0.5^2 - 0.2^2) - (50 - 10)\right]\frac{ln\frac{r}{0.2}}{ln\frac{0.5}{0.2}} + 50$

$$T = -1000(r^2 - (0.2)^2) + 185.53ln\frac{r}{0.2} + 50$$

- To find the temperature at mid-point of the wall

$r_m = \frac{r_i + r_o}{2} = \frac{0.2+0.5}{2} = 0.35m$

$$T = -1000(r^2 - (0.2)^2) + 185.53 ln\frac{r}{0.2} + 50$$

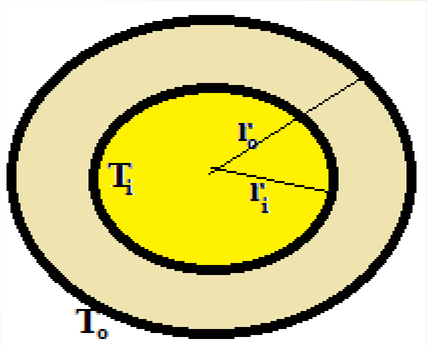
- $T_m = -1000\left((0.35)^2 - (0.2)^2\right) + 185.53 ln\frac{0.35}{0.2} + 50 = 71.33^o C$
- To find the location of maximum temperature
- $r = \left\{\frac{2k}{\ddot{g}}\left[\frac{\ddot{g}}{4k}\left(r_o^2 - r_i^2\right) - (T_i - T_o)\right]\frac{1}{ln\frac{r_o}{r_i}}\right\}^{1/2}$
- $r = \left\{\frac{2\times 10}{4\times 10^4}\left[\frac{4\times 10^4}{4k}(0.5^2 - 0.2^2) - (50 - 10)\right]\frac{1}{ln\frac{0.5}{0.2}}\right\}^{1/2}$
- $r = 0.3045m$
- $T_{max} = -1000\left((0.3045)^2 - 0.2^2\right) + 185.53 ln\frac{0.3045}{0.2} + 50$
  $= 80.71^o C$

- To find the heat flux at the inner surface and outer surface, we use:
- $q_{r=r_i} = -k\frac{dT}{dr})_{r=r_i}$
- $q_{r=r_i} = -k\left\{-\frac{\ddot{g}}{4k}(2r) + \left[\frac{\ddot{g}}{4k}\left(r_o^2 - r_i^2\right) - (T_i - T_o)\right]\frac{1}{r\,ln\left(\frac{r_o}{r_i}\right)}\right\}$
- $q_{r=r_i} = -k\left\{-\frac{\ddot{g}}{4k}(2r_i) + \left[\frac{\ddot{g}}{4k}\left(r_o^2 - r_i^2\right) - (T_i - T_o)\right]\frac{1}{r_i\,ln\left(\frac{r_o}{r_i}\right)}\right\}$

- $q_{r=r_i}$
$$= -10\left\{-\frac{4\times10^4}{4\times10}(2\times0.2)+\left[\frac{4\times10^4}{4\times10}(0.5^2-0.2^2)-(50-10)\right]\frac{1}{0.2\ ln\left(\frac{0.5}{0.2}\right)}\right\}$$
$$= -1281.37W$$
- $$q_{r=r_o} = -k\left\{-\frac{\ddot{g}}{4k}(2r_o)+\left[\frac{\ddot{g}}{4k}\left(r_o^2-r_i^2\right)-(T_i-T_o)\right]\frac{1}{r_o\ ln\left(\frac{r_o}{r_i}\right)}\right\}$$
- $q_{r=r_o}$
$$= -10\left\{-\frac{4\times10^4}{4\times10}(2\times0.5)+\left[\frac{4\times10^4}{4\times10}(0.5^2-0.2^2)-(50-10)\right]\frac{1}{0.5\ ln\left(\frac{0.5}{0.2}\right)}\right\}$$
$$= 8943.73W$$

Heat Conduction in Spherical Body

- Hollow Spherical body without heat generation
- $\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right) = 0$ If k=constant then
- $\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = 0$
- The Boundary conditions are
- B.C.1 At $r = r_i$ $T = T_i$
- B.C.2 At $r = r_o$ $T = T_o$



- The differential equation $\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0$
- It can be integrated first integration
- $r^2 \frac{dT}{dr} = C_1$ or $\frac{dT}{dr} = C_1 r^{-2}$
- By second integrating in becomes
- $T = -C_1 r^{-1} + C_2$ or $T = \frac{C_1}{r} + C_2$ T.D.E
- By substituting the boundary conditions

From B.C.1 $T_i = \frac{C_1}{r_i} + C_2$ (1)

from B.C.2 $T_o = \frac{C_1}{r_o} + C_2$ (2)

By substituting eq.(2) from eq.(1)

$$T_i - T_o = \frac{C_1}{r_i} - \frac{C_1}{r_o} = C_1\left(\frac{1}{r_i} - \frac{1}{r_o}\right)$$

$C_1 = \frac{(T_i - T_o)}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}$ by substituting this in eq.(1) we get

$T_i = \frac{(T_i - T_o)}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\frac{1}{r_i} + C_2$ then

$$C_2 = T_i - \frac{(T_i - T_o)}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\frac{1}{r_i}$$

By substituting C_1 & C_2 in T.D.E we get

- $$T = \frac{(T_i - T_o)}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\frac{1}{r} + T_i - \frac{(T_i - T_o)}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\frac{1}{r_i}$$ or
- $$T - T_i = (T_i - T_o)\frac{\left(\frac{1}{r} - \frac{1}{r_i}\right)}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}$$ or
- $$\frac{(T - T_i)}{(T_o - T_i)} = \frac{\left(\frac{1}{r_i} - \frac{1}{r}\right)}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}$$ T.D.E

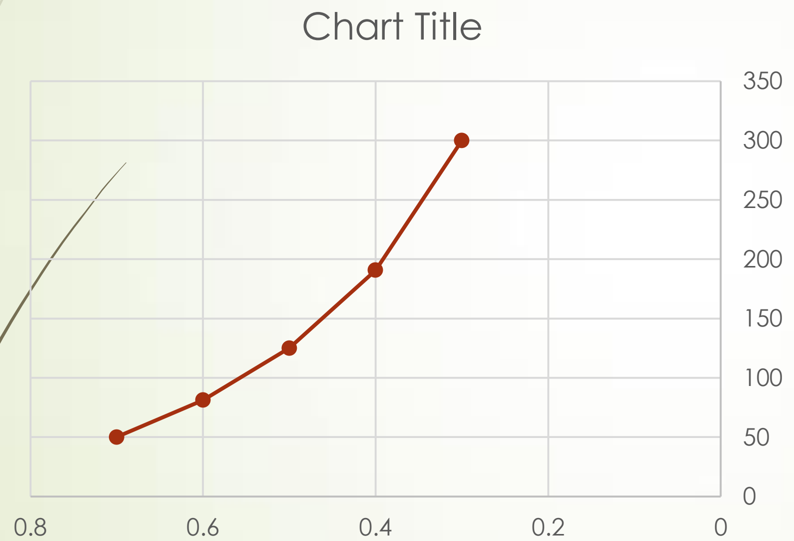
- **Example:** **hollow spherical wall of inside radius 0.3m and outside radius is 0.7m. Thermal conductivity of its material is 12W/m.^{o}C. The temperature of inner surface is $300^{o}C$ and temperature of outer surface is $50^{o}C$. Find the T.D. E. and find the temperature magnitude at ¼ , ½ and ¾ of the thickness of the shell. Determine also the heat transfer form the spherical shell.**
- **Solution:** spherical shell of $r_i = 0.3m\ and\ r_o = 0.7m$ with $T_i = 300^{o}C$ and $T_o = 50^{o}C$ and k=12W/m.^{o}C.
- **Assumption:** one-dimensional heat conduction with constant thermal conductivity.

- **Analysis:** the differential equation is $\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0$

- The solution of this equation
- $\frac{(T-T_i)}{(T_o-T_i)}=\frac{\left(\frac{1}{r_i}-\frac{1}{r}\right)}{\left(\frac{1}{r_i}-\frac{1}{r_o}\right)}$
- $\frac{(T-300)}{(50-300)}=\frac{\left(\frac{1}{0.3}-\frac{1}{r}\right)}{\left(\frac{1}{0.3}-\frac{1}{0.7}\right)}\rightarrow\frac{(T-300)}{(-250)}=\frac{\left(\frac{1}{0.3}-\frac{1}{r}\right)}{(1.90476)}$
- At ¼ of the shell r=0.4m $T=190.62^oC$
- At ½ of the shell r=0.5m $T=125^oC$
- At ¾ of the shell r=0.6m $T=81.25^oC$

- To find the heat transfer from the spherical shell

$$Q = \frac{(T_i - T_o)}{\frac{1}{4\pi k}\left(\frac{1}{r_i} - \frac{1}{r_o}\right)} = \frac{(300-50)}{\frac{1}{4\pi \times 12}\left(\frac{1}{0.3} - \frac{1}{0.7}\right)} = 19792W$$



T	r
300	0.3
190.62	0.4
125	0.5
81.25	0.6
50	0.7

Heat conduction in solid Sphere with heat generation

- Solid sphere with outer radius R and constant thermal conductivity k. The heat generation in the sphere is $\ddot{g} W/m^3$. To find the temperature distribution through the solid sphere. The Boundary conditions are
- At r=0 $\frac{dT}{dr} = 0$, at r=R $T = T_o$.
- The differential equation of temperature distribution is

- $\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{\ddot{g}r^2}{k}$
- By integrating this equation we get that:
- $r^2\frac{dT}{dr} = -\frac{\ddot{g}r^3}{3k} + C_1$
- From B.C.1 where r=0 $\frac{dT}{dr} = 0$ then $C_1 = 0$
- $r^2\frac{dT}{dr} = -\frac{\ddot{g}r^3}{3k} \quad \rightarrow \quad \frac{dT}{dr} = -\frac{\ddot{g}r}{3k}$
- By integrating this equation we get that
- $T = -\frac{\ddot{g}r^2}{6k} + C_2$ From B.C.2 where r=R T=T_R
- $C_2 = \frac{\ddot{g}R^2}{6k} + T_R$

$$\boldsymbol{T = -\frac{\ddot{g}r^2}{6k} + \frac{\ddot{g}R^2}{6k} + T_R}$$ T.D.E

- $T - T_R = \frac{\ddot{g}R^2}{6k}\left(1 - \left(\frac{r}{R}\right)^2\right)$ (1)
- The temperature at the center of the sphere is
- $T_o - T_R = \frac{\ddot{g}R^2}{6k}\left(1 - \left(\frac{0}{R}\right)^2\right) = \frac{\ddot{g}R^2}{6k}$ (2)
- By dividing eq.(1) by eq.(2) we get:
- $\boldsymbol{\frac{T-T_R}{T_o-T_R} = \frac{\frac{\ddot{g}R^2}{6k}\left(1-\left(\frac{r}{R}\right)^2\right)}{\frac{\ddot{g}R^2}{6k}} = \left(1 - \left(\frac{r}{R}\right)^2\right)}$

- To find the heat flux at the outer surface of the sphere, we do that:
- $q_R = -k\frac{dT}{dr})_{r=R} = -k\frac{d}{dr}\left(-\frac{\ddot{g}r^2}{6k} + \frac{\ddot{g}R^2}{6k} + T_R\right)$
- $q_R = -k\left(-\frac{2\ddot{g}r}{6k}\right)_{r=R} = \frac{\ddot{g}R}{3}\ W/m^2$
- $\dot{Q}_R = Aq_R = 4\pi R^2\frac{\ddot{g}R}{3} = \frac{4}{3}\pi R^3\ddot{g}$

Heat Conduction in a Hollow sphere with heat generation

- Hollow sphere with inner radius r_i and outer radius r_o, thermal conductivity of its material is k and the heat generation is $\ddot{g}$. It is to find the temperature distribution equation.
- The differential equation is $\frac{d}{dr}\left(r^2\frac{dT}{dr}\right) = -\frac{\ddot{g}r^2}{k}$
- The Boundary conditions are:
- B.C.1 at r=r_i $T = T_i$ B.C.2 r=r_o $T = T_o$

By integrating the equation we get:

- $$r^2 \frac{dT}{dr} = -\frac{\ddot{g} r^3}{3k} + C_1$$
- $\frac{dT}{dr} = -\frac{\ddot{g} r}{3k} + \frac{C_1}{r^2}$ and by second integration
- $T = -\frac{\ddot{g} r^2}{6k} - \frac{C_1}{r} + C_2$ T.D.E.
- By applying the boundary conditions
- B.C.1 $T_i = -\frac{\ddot{g} r_i^2}{6k} + \frac{C_1}{r_i} + C_2$ (1)
- B.C.2 $T_o = -\frac{\ddot{g} r_o^2}{6k} + \frac{C_1}{r_o} + C_2$ (2)
- By subtracting eq.(2) from eq.(1) we get:

- $$T_i - T_o = \frac{\ddot{g}}{6k}\left(r_o^2 - r_i^2\right) + C_1\left(\frac{1}{r_i} - \frac{1}{r_o}\right)$$
- $$C_1 = \frac{1}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\left[\frac{\ddot{g}}{6k}\left(r_o^2 - r_i^2\right) - \left(T_i - T_o\right)\right]$$
- By substituting in eq.(1) we get:
- $$T_i = -\frac{\ddot{g} r_i^2}{6k} + \frac{1}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\left[\frac{\ddot{g}}{6k}\left(r_o^2 - r_i^2\right) - \left(T_i - T_o\right)\right]\frac{1}{r_i} + C_2$$
- $$C_2 = T_i + \frac{\ddot{g} r_i^2}{6k} - \frac{1}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\left[\frac{\ddot{g}}{6k}\left(r_o^2 - r_i^2\right) - \left(T_i - T_o\right)\right]\frac{1}{r_i}$$
- $$T = -\frac{\ddot{g} r^2}{6k} - \frac{1}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\left[\frac{\ddot{g}}{6k}\left(r_o^2 - r_i^2\right) - \left(T_i - T_o\right)\right]\frac{1}{r} + T_i + \frac{\ddot{g} r_i^2}{6k} + \frac{1}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\left[\frac{\ddot{g}}{6k}\left(r_o^2 - r_i^2\right) - \left(T_i - T_o\right)\right]\frac{1}{r_i}$$
- $$\boldsymbol{T = -\frac{\ddot{g}}{6k}\left(r^2 - r_i^2\right) + \frac{1}{\left(\frac{1}{r_i} - \frac{1}{r_o}\right)}\left[\frac{\ddot{g}}{6k}\left(r_o^2 - r_i^2\right) - \left(T_i - T_o\right)\right]\left(\frac{1}{r_i} - \frac{1}{r}\right) + T_i}$$