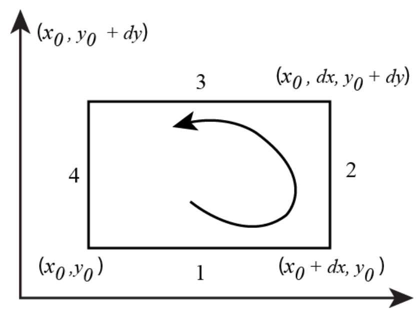
Lecture 4

Curl

The curl of a vector field gives a measure of its local vorticity or the rotation. Suppose a floatable material is placed on the surface of water. If the material rotates, then the velocity flow has a curl, the magnitude of which is demonstrated by how fast it rotates.

Physical Significance of Curl

To demonstrate the operation mathematically, let us consider circulation of a fluid around a differential loop in the $x - y$ plane as shown in the figure. The sides of the loop are dx and dy



The circulation of the fluid, τ is defined as $\int_C \vec{v}.d\vec{l}$ where $\vec{v}$ is the velocity of the fluid and C denotes the path in which the fluid circulates. The circulation around the loop in an anticlockwise fashion is given by,

$$\tau_{1234} = \int_1 v_x\ (x,y)dl_x + \int_2 v_y(x,y)\ dly + \int_3 v_x(x,y)dlx + \int_4 v_y\ (x,y)dly$$

The paths 1, 2, 3, 4 are shown in figure. Around path 1, dl_x is positive, but for path 3, dl_x is negative. Similarly for path 2, dl_y is positive, while for path 4, it is negative.

Thus,

$$\tau_{1234} = v_x\ (x_0, y_0)dx + [v_y(x_0, y_0)dx + \left[v_y(x_0, y_0) + \frac{dv}{dx}dx\right]dy + \left[v_x\ (x_0, y_0) + \frac{dv_x}{dy}dy\right](-dx) + v_y(x_0, y_0)\ (-dy)$$

$$= (\frac{dv_y}{dx} - \frac{dv_x}{dy})dxdy = (\vec{\nabla} \times \vec{v})_z\ dxdy$$

where $(\vec{\nabla} \times \vec{v})_z = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$

Thus, circulation per unit area of the loop is $(\frac{dv_y}{dx} - \frac{dv_x}{dy})$. In a three dimensional sense, that is including $(\vec{\nabla} \times \vec{v})_x$ and $(\vec{\nabla} \times \vec{v})_y$ the circulation is $\vec{\nabla} \times \vec{v}$.

In a compact form, the curl is represented by,

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

Some interesting properties

1) $\vec{\nabla}.(\vec{\nabla} \times \vec{A}) = 0$: a vector that curls, does not have a divergence
2) $\vec{\nabla} \times (\vec{\nabla}\varphi) = 0$: gradient of a scalar is a fixed direction in space. It does not curl

Example Consider a vector field,

$$\vec{v} = y\,\hat{x} - x\,\hat{y}$$

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\,\hat{k} \text{ every where in space.}$$

Example

Given : $\vec{\nabla} \times (\overrightarrow{\varphi A}) = \varphi\,\vec{\nabla} \times \vec{A} + \vec{\nabla}\varphi \times \vec{A}$

Where φ and $\vec{A}$ are arbitrary scalar and vectors respectively. Using the above relation find the value of $\vec{\nabla} \times \left(\frac{1}{r}\right)$.

Solution

$\vec{\nabla} \times \left(\frac{1}{r}\right) = \vec{\nabla} \times (\frac{\vec{r}}{r^2})$

So $\varphi = \frac{1}{r^2}$, $\vec{A} = \vec{r}$

Using the above relation,

$\nabla \times \left(\frac{1}{r}\right) = \frac{1}{r^2}\,\vec{\nabla} \times \vec{r} + \vec{\nabla}\left(\frac{1}{r^2}\right) \times \vec{r}$

The first term on the right is zero as $\vec{r}$ is curl-less vector.

$$\vec{\nabla}\left(\frac{1}{r^2}\right) = -\frac{2\vec{r}}{(x^2+y^2+z^2)}$$

$$\vec{\nabla}\left(\frac{1}{r^2}\right) x\vec{y} = \frac{-2\vec{r} \times \vec{r}}{(x^2+y^2+z^2)} = 0$$

$$\text{Thus } \vec{\nabla} \times \left(\frac{1}{r}\right) = 0$$

If the curl of a vector field is zero, then the force field is known to be conservative.

Example

A force field is given by,

$$\vec{F} = (x + 2y + \alpha z)\,\hat{i} + (\beta x - 3y - z)\,\hat{j} + (4x + \gamma y + 2z)\,\hat{k}$$

For what choices α, β and γ, $\vec{F}$ is conservative?

Solution $\vec{\nabla} \times \vec{F} = 0$

Yields, $\alpha = 4, \beta = 2, \gamma = -1$

Reader may please check it.

Note : For every conservative force, there exists a potential function, *v* such that $\vec{F} = -\vec{\nabla}V$. Can you find the potential *V* for this case?

Vector Integration

We shall close the discussion on vector calculus by discussing integration and some important theorems. The three integrals that we are concerned about are (a) line integral, (b) surface integral, (c) volume integral.

(a) Line integral

Consider the integral of the type

$\int_c \vec{F}(\vec{r}).d\vec{r} = \int_c (F_x dx + F_y\, dy + F_z dz)$

Where *c* is a path connecting two points A and B. For a conservative force, that is, $\vec{\nabla} \times \vec{F} = 0$, the integral is independent of the path c and depends on the values at the end points. The proof can be given as follows.

$\vec{\nabla}\times\vec{F}=0$ implies that $\vec{F}$ can be written as
$\vec{F}=\vec{\nabla}V$ (neglecting the negative sign)

$$So,\ \ F_x=\frac{dV}{dx}\ ,F_y=\frac{dV}{dy}\ ,F_z=\frac{dV}{dz}$$

$$\int_A^B \vec{F}\,.\,d\vec{r}=\int_A^B (F_x\,dx+F_y dy+F_z dz)=\int_A^B (\frac{dV}{dx}dx+\frac{dV}{dy}dy+\frac{dV}{dz}dz)$$

$$=\int_A^B (\frac{dV}{dx}\,\frac{dx}{dl}+\frac{dV}{dy}\,\frac{dy}{dl}+\frac{dV}{dz}\,\frac{dz}{dl})\,dl=\int_A^B \frac{dV}{dl}dl=V(B)-V(A)$$

(b) Surface integral

A surface integral of a vector field, $\vec{F}$ is defined by
$\int_S \vec{F}\,.\,d\vec{S}=\int_S \vec{F}.\hat{n}\,ds$ where the direction of the elemental area is along the outward drawn normal.

Example

Consider a vector field given by,
$\vec{F}=z\hat{i}+x\,\hat{j}-3y^2z\,\hat{k}$. Find the flux of this field through the curved surface of a cylinder, $x^2+y^2=16$ included in the first octant between z = 0 and z = 5.

$$\int_s \vec{F}\,.\,\hat{n}\,ds=\int_s \vec{F}\,.\,\hat{n}\,\frac{dxdz}{\left|\hat{n}.\hat{j}\right|}$$

$$\hat{n}=\frac{\vec{\nabla}(\mathrm{x}^2+\mathrm{y}^2)}{\left|\vec{\nabla}(\mathrm{x}^2+\mathrm{y}^2)\right|}=2\frac{x\hat{i}+y\,\hat{j}}{4}$$

$$\vec{A}.\,\hat{n}=\frac{1}{4}(xz+xy)$$

$$\hat{n}.\ \hat{j}=\frac{y}{4}=\sqrt{\frac{16-x^2}{4}}$$

$$\int_s \vec{F}\,.\,\hat{n}\,ds=\int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{16-x^2}}+x\right)dxdz$$
$$=90$$

A surface integral can also be done for a scalar field. In an example below we show how it can be done.

Example

Find the moment of inertia I of homogeneous spherical lamina of radius r and mass M about z axis.

Solution

$I = \int_S \sigma z^2 ds$ where $\sigma = \frac{M}{A}$ mass density.

Using $z = r\cos\theta$ and $ds = r^2 dr \cos\theta d\theta$.

Putting everything together and using $A = 4\pi a^2$

$I = \frac{2Mr^2}{3}$

(c) Volume integral

Having discussed the line and surface integrals, we need to talk about the volume integrals. As can be seen immediately afterwards, we shall need the volume integral of a scalar function.

The volume integral is denoted by,

$$\int f(x, y, z) dxdydz$$

Nevertheless, the volume integral can also be defined for a vector field in a similar manner, namely

$$\int \vec{A}(x, y, z) dxdydz$$

Example

Evaluate $\int_v (2x + y) dv$ where v is the closed region bound by the cylinder $z = u - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$, $z = 0$.

$$\int_{x=0}^{2} \int_0^2 \int_{z=0}^{u-x^2} (2x + y) dxdydz) = \int_0^2 \int_0^2 (2x + y)(u - x^2) dxdy$$

$$= \int_0^2 \int_0^2 (8x - 2x^3 + 4y + yx^2) dxdy$$

$$= \frac{80}{3}$$