جامعة المستقبل
AL MUSTAQBAL UNIVERSITY

كلية العلوم
قسم الانظمة الطبية الذكية

POLAR COORDINATES

Mathematics

المرحلة الاولى

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المحاضرة الثانية

Polar Coordinates

Let r and θ be polar coordinates of the point (x, y) corresponding to a nonzero complex number $z = x + iy$

Since, $x = r\cos\theta$, $y = r\sin\theta$

$\boxed{z = r(\cos\theta + i\sin\theta)}$, *polar form of number z*

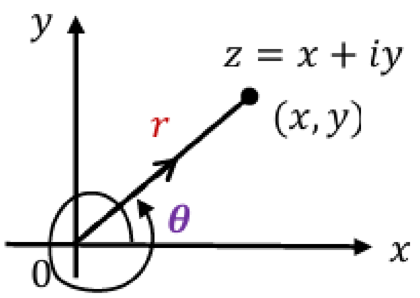
For example,

$$1 + i = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\cos\left(\frac{-7\pi}{4}\right) + i\sin\left(\frac{-7\pi}{4}\right)\right]$$

The number θ is called an *argument* of z, (also called, *'phase' and 'angle'*)

θ is the directed angle measured from the positive x-axis to a point p and called (*argument* of z) and is denoted by : arg ; $\theta = \arg z_1$

Hence the angles will be measured in *radian* and positioned in the *counterclockwise sense*. $\boxed{\tan\theta = \frac{y}{x}}$



Important identity

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
2. $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

Question : Find the value of (iz)

$$iz = \underbrace{(1)\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]}_{i}\underbrace{r(\cos\theta + i\sin\theta)}_{z}$$

$$iz = r\left[\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right]$$

iz has an important applications in Optics i.e. (*circular polarization*).

Example:

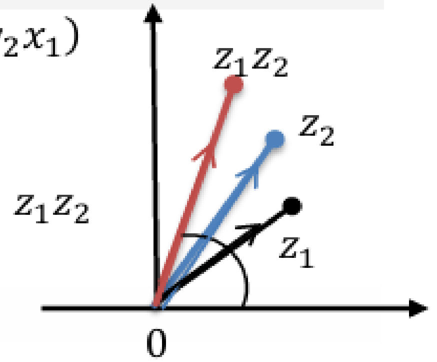
$\theta_1 = \arg(1+i)$, $\theta_1 = arg(-1-i) \Longrightarrow \tan\theta_1 = \tan\theta_2 = 1$….**PLOT!!**

Question : Use Cartesian coordinate system representation of z , simplified the multiplication of two different complex numbers.

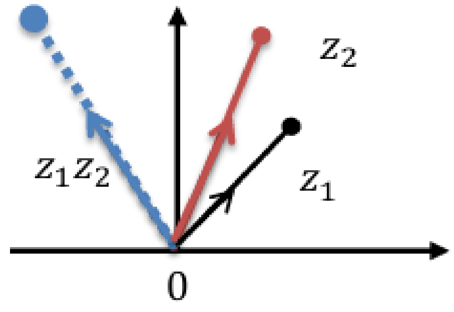
$$z_1 z_2 = (x_1 + iy_2)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + y_2 x_1)$$

$$(1 + i)(2 + 3i) = 2 + 5i - 3 = -1 + 5i$$

$z_1 = (1 + i) \rightarrow x_1 = 1, y_1 = 1, \ r_1 = \sqrt{2}, \ \theta_1 = 45°$,



$\tan^{-1}\frac{3}{2}$ or $\theta_2 = 56.31°$,



For the ***final results*** $-1 + 5i$, will be :

$r = 5.1$, $\theta = \tan^{-1} -\frac{5}{1}$, $\theta = 101.31°$

Exponential Function الدالة الاسية العقدية

The polar form is $z = r(\cos\theta + i\sin\theta)$,

or , $z = re^{i\theta}$

Where $\boxed{e^{i\theta} = (\cos\theta + i\sin\theta)}$, is ***Euler's formula*** , and $e^{i\theta}$ (*complex exponential function*)

NOTE:

$$(1)\ z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$(2)\ \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Question **: Relate trigonometric function with complex exponential function**

الربط بين الدوال الاسية العقدية والدوال المثلثية الحقيقة

Proof: $\boxed{e^{-i\theta} = \cos\theta - i\sin\theta}$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \dots (1)$$

$$\underline{e^{-i\theta} = \cos\theta - i\sin\theta} \quad \dots (2)$$

Adding (1) and (2) : $e^{i\theta} + e^{-i\theta} = 2\cos\theta \Longrightarrow \boxed{\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}}$

subtracting (1) and (2)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Powers of complex numbers

$$\boxed{z^n = r^n e^{in\theta}},\quad (n = 0, \pm 1, \pm 2, \dots) \quad \text{, where } \left(e^{i\theta}\right)^n = e^{in\theta}$$

Or $$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = \cos\phi + i\sin\phi},$$

$$(n = 0, \pm 1, \pm 2, \dots) \quad (\textbf{de Moivre's theorem!!})$$

Where $\boxed{\phi = \boldsymbol{n}\theta}$

Roots of complex number z

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right], \qquad \boxed{k = 0,1,2,\dots,n-1}$$

Example

1. Solve the equation $z^3 + 1 = 0$, or in other words
2. Find z for the equation $z = (-1)^{\frac{1}{3}}$
3. Find the third roots(***ONLY***) of the equation $z^3 + 1 = 0$

Solution

$$z^3 = -1, = (-1)^{\frac{1}{3}},$$

$$x = -1, y = 0, r = \sqrt{x^2 + y^2} \Longrightarrow r = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Longrightarrow \quad \theta = \pi, \qquad n = 3, \qquad k = 0,1,2$$

$1^{st}\ root\ \boxed{k = 0},\ z_1 = (1)\left[cos\frac{\pi+2(0)\pi}{3} + i\,sin\frac{\pi+2(0)\pi}{3}\right]$

$$= \frac{1}{2} + i0.866 = 0.5 + i0.866$$

$2^{nd}\ root\ \boxed{k = 1},$

$$z_2 = \left[cos\frac{\pi + 2\pi}{3} + i\,sin\frac{\pi + 2\pi}{3}\right]$$

$$= \cos\pi + i\sin\pi = -1 + 0i$$

$3^{rd}\ root\ \boxed{k = 2},$

$$z_3 = \left[cos\frac{\pi + 4\pi}{3} + i\,sin\frac{\pi + 4\pi}{3}\right]$$

$$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = 0.5 - 0.866i$$

The 3^{rd} root $k = 2$ was $(0.5\ , -0.866)$

Examples

1. Find one value of arg(z) when

$$\textbf{a)}\ z=\frac{-2}{1+i\sqrt{3}}\ ,\qquad \textbf{b)}\ z=\frac{i}{-2-2i}\ ,\qquad \textbf{c)}\ z=\left(\sqrt{3}-i\right)^{6}$$

Solution:

$$\textbf{a)}\ z=\frac{-2}{1+i\sqrt{3}}\ ,\qquad \frac{-2}{1+i\sqrt{3}}\ \frac{1-i\sqrt{3}}{1-i\sqrt{3}}=\frac{(-2)\left(1-\sqrt{3}i\right)}{1+3}$$

$$=\frac{-2+2\sqrt{3}i}{4}=-\frac{1}{2}+\frac{1}{2}\sqrt{3}i$$

$$\therefore\ \boldsymbol{x}=-\frac{\mathbf{1}}{\mathbf{2}}=-\mathbf{0.5}$$

$$\boldsymbol{y}=\frac{\mathbf{1}}{\mathbf{2}}\sqrt{\mathbf{3}}=\mathbf{0.866}\Longrightarrow\theta=\arg(z)=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{{}^{1}/_{2}\sqrt{3}}{/}\right)$$

$$\theta=\tan^{-1}\ (-\sqrt{3})\Longrightarrow\boxed{\theta=-60},\theta=180^{\circ}-60^{\circ}=120^{\circ}$$

$$\textbf{b)}\ \ \frac{i}{-z-2i}\ ,$$

$$\frac{i}{-z-2i}=\frac{i}{-2-2i}*\frac{-2+2i}{-2+2i}=\ \frac{i(-2+2i)}{8}$$

$$\frac{i}{-z-2i} = \frac{-2i+2}{8}$$

$$\frac{i}{-z-2i} = \frac{1-i}{4} = \frac{\mathbf{1}}{\mathbf{4}} - \frac{\mathbf{1}}{\mathbf{4}} i$$

$$x = \frac{1}{4}, y = -\frac{1}{4} \Longrightarrow \theta = \tan^{-1}\left(\frac{-1/4}{/}\right) = \tan^{-1}(-1) = -45° = 315° = -\frac{\pi}{4} \equiv \frac{3\pi}{4}$$

c) $z = \left(\sqrt{3} - i\right)^6$

$$x = \sqrt{3}$$

$$y = -1, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -30°$$

$$\boldsymbol{n\theta} = 6 * \theta = -30° * 6 = -180° = 360° - 180° = \pi$$