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جامعة المستقبل
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كلية العلــــوم
قســــــــم الانــظــمــة الـــطبيـة الـــذكـــيـــة

المحاضرة الخامسة

INTEGRATION

المادة : الرياضيات
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Integration

The idea of integration is that we can compute many quantities by breaking them into small pieces, and then summing the contribution from each small part.

1. Indefinite integrals:

The set of all anti derivatives of a function is called indefinite integral of the function. Assume u and v denote differentiable functions of x, and a, n, and c are constants, then the integration formulas are:-

$$1)\ \int du = u(x) + c$$

$$2)\ \int a \cdot u(x)dx = a\int u(x)dx$$

$$3)\ \int \left(u(x) \mp v(x)\right)dx = \int u(x)\,dx \mp \int v(x)\,dx$$

$$4)\ \int u^n du = \frac{u^{n+1}}{n+1} + c \quad when \quad n \neq -1 \quad \& \quad \int u^{-1} du = \int \frac{1}{u} du = \ln u + c$$

$$5)\ \int a^u du = \frac{a^u}{\ln a} + c \qquad \Rightarrow \qquad \int e^u du = e^u + c$$

EX-1 – **Evaluate the following integrals:**

$1)\ \int 3x^2\, dx$

$2)\ \int \left(\frac{1}{x^2} + x\right) dx$

$3)\ \int x\sqrt{x^2+1}\, dx$

$4)\ \int \left(2t + t^{-1}\right)^2 dt$

$5)\ \int \sqrt{(z^2 - z^{-2})^2 + 4}\, dz$

$6)\ \int \frac{x+3}{\sqrt{x^2+6x}}\, dx$

$7)\ \int \frac{x+2}{x^2}\, dx$

$8)\ \int \frac{e^x}{1+3e^x}\, dx$

$9)\ \int 3x^3 \cdot e^{-2x^4}\, dx$

$10)\ \int 2^{-4x}\, dx$

Sol. –

$$1)\ \int 3x^2\, dx = 3\int x^2 dx = 3\frac{x^3}{3} + c = x^3 + c$$

$$2)\ \left(x^{-2}+x\right)dx=\int x^{-2}\,dx+\int x\,dx=\frac{x^{-1}}{-1}+\frac{x^{2}}{2}+c=-\frac{1}{x}+\frac{x^{2}}{2}+c$$

$$3)\ \int x\sqrt{x^{2}+1}\,dx=\frac{1}{2}\int 2x(x^{2}+1)^{1/2}dx=\frac{1}{2}\frac{(x^{2}+1)^{3/2}}{3/2}+c=\frac{1}{3}\sqrt{(x^{2}+1)3}+c$$

$$4)\ \int\left(2t+t^{-1}\right)^{2}dt=\int\left(4t^{2}+4+t^{-2}\right)dt=4\frac{t^{3}}{3}+4t+\frac{t^{-1}}{-1}+c=\frac{4}{3}t^{3}+4t-\frac{1}{t}+c$$

$$5)\ \int\sqrt{(z^{2}-z^{-2})^{2}+4}\,dz=\int\sqrt{z^{4}-2+z^{-4}+4}\,dz=\int\sqrt{z^{4}+2+z^{-4}}\,dz$$

$$=\int\sqrt{(z^{2}+z^{-2})^{2}}\,dz=\int(z^{2}+z^{-2})\,dz=\frac{z^{3}}{3}+\frac{z^{-1}}{-1}+c=\frac{1}{3}z^{3}-\frac{1}{z}+c$$

$$6)\ \int\frac{x+3}{\sqrt{x^{2}+6x}}\,dx=\frac{1}{2}\int(2x+6)\cdot(x^{2}+6x)^{-1/2}dx$$

$$=\frac{1}{2}\cdot\frac{(x^{2}+6x)^{1/2}}{1/2}+c=\sqrt{x^{2}+6x}+c$$

$$7)\ \int\frac{x+2}{x^{2}}\,dx=\int\left(\frac{x}{x^{2}}+\frac{2}{x^{2}}\right)dx=\int\left(x^{-1}+2x^{-2}\right)dx=\ln x+\frac{2x^{-1}}{-1}+c=\ln x-\frac{2}{x}+c$$

$$8)\ \int\frac{e^{x}}{1+3e^{x}}\,dx=\frac{1}{3}\int 3e^{x}(1+3e^{x})^{-1}dx=\frac{1}{3}\ln(1+3e^{x})+c$$

$$9)\ \int 3x^{3}\cdot e^{-2x^{4}}\,dx=-\frac{3}{8}\int-8x^{3}\cdot e^{-2x^{4}}\,dx=-\frac{3}{8}\cdot e^{-2x^{4}}+c$$

$$10)\ \int 2^{-4x}dx=-\frac{1}{4}\int 2^{-4x}\cdot(-4dx)=-\frac{1}{4}\cdot 2^{-4x}\cdot\frac{1}{\ln 2}+c$$

2. Integrals of trigonometric functions:

The integration formulas for the trigonometric functions are:

$$6)\ \int\sin u\cdot du=-\cos u+c$$

$$7)\ \int\cos u\cdot du=\sin u+c$$

$$8)\ \int\tan u\cdot du=-\ln|\cos u|+c$$

$$9)\ \int\cot u\cdot du=\ln|\sin u|+c$$

$10)\ \int \sec u \cdot du = \ln|\sec u + \tan u| + c$

$11)\ \int \csc u \cdot du = -\ln|\csc u + \cot u| + c$

$12)\ \int \sec^2 u \cdot du = \tan u + c$

$13)\ \int \csc^2 u \cdot du = -\cot u + c$

$14)\ \int \sec u \cdot \tan u \cdot du = \sec u + c$

$15)\ \int \csc u \cdot \cot u \cdot du = -\csc u + c$

Example 2: **Evaluate the following integrals:**

$1)\ \int \cos(3\theta - 1)d\theta$

$2)\ \int x \cdot \sin(2x^2)\,dx$

$3)\ \int \cos^2(2y) \cdot \sin(2y)\,dy$

$4)\ \int \sec^3 x \cdot \tan x\ \ dx$

$5)\ \int \sqrt{2 + \sin 3t} \cdot \cos 3t\ \ dt$

$6)\ \int \frac{d\theta}{\cos^2\theta}$

$7)\ \int (1 - \sin^2 3t) \cdot \cos 3t\ \ dt$

$8)\ \int \tan^3(5x) \cdot \sec^2(5x)\ dx$

$9)\ \int \sin^4 x \cdot \cos^3 x\ \ dx$

$10)\ \int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx$

Solution:

$$1)\ \frac{1}{3}\int 3\cos(3\theta - 1)d\theta = \frac{1}{3}\sin(3\theta - 1) + c$$

$$2)\ \frac{1}{4}\int 4x \cdot \sin(2x^2)dx = -\frac{1}{4}\cos(2x^2) + c$$

$$3)\ -\frac{1}{2}\int (\cos 2y)^2 \cdot (-2\sin 2y\ \ dy) = -\frac{1}{2} \cdot \frac{(\cos 2y)^3}{3} + c = -\frac{1}{6}(\cos 2y)^3 + c$$

$$4)\ \int \sec^2 x \cdot (\sec x \cdot \tan x \cdot dx) = \frac{\sec^3 x}{3} + c$$

$$5)\ \frac{1}{3}\int (2 + \sin 3t)^{1/2}(3\cos 3t\ \ dt) = \frac{1}{3} \cdot \frac{(2 + \sin 3t)^{3/2}}{3/2} + c = \frac{2}{9}\sqrt{(2 + \sin 3t)^3} + c$$

$$6)\int \frac{d\theta}{\cos^2\theta} = \int \sec^2\theta \cdot d\theta = \tan\theta + c$$

$$7)\int (1-\sin^2 3t)\cdot \cos 3t \;\; dt = \frac{1}{3}\int 3\cos 3t \;\; dt - \frac{1}{3}\int (\sin 3t)^2 \cdot 3\cos 3t \;\; dt$$

$$= \frac{1}{3}\sin 3t - \frac{1}{3}\cdot\frac{\sin^3 3t}{3} + c = \frac{1}{3}\cdot\sin 3t - \frac{1}{9}\sin^3 3t + c$$

$$8)\ \frac{1}{5}\int \tan^3 5x\cdot(5\sec^2 5x \;\; dx) = \frac{1}{5}\cdot\frac{\tan^4 5x}{4} + c = \frac{1}{20}\tan^4 5x + c$$

$$9)\int \sin^4 x\cdot\cos^3 x \;\; dx = \int \sin^4 x\cdot(1-\sin^2 x)\cdot\cos x \;\; dx$$

$$= \int \sin^4 x\cdot\cos x \;\; dx - \int \sin^6 x\cdot\cos x \;\; dx = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + c$$

$$10)\int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\csc^2\sqrt{x}-1}{\sqrt{x}}\,dx = 2\int \frac{\csc^2\sqrt{x}}{2\sqrt{x}} - \int x^{-1/2}\,dx$$

$$= 2(-\cot\sqrt{x}) - \frac{x^{1/2}}{1/2} + c = -2\cot\sqrt{x} - 2\sqrt{x} + c$$

3. Integrals of inverse trigonometric functions:

The integration formulas for the inverse trigonometric functions are:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c \qquad ; \qquad \forall u^2 < a^2$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c = -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| + c \qquad ; \qquad \forall u^2 > a^2$$

Example 3: **Evaluate the following integrals:**

1) $\int \frac{x^2}{\sqrt{1-x^6}}dx$

2) $\int \frac{dx}{\sqrt{9-x^2}}$

3) $\int \frac{x}{1+x^4}dx$

4) $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx$

5) $\int \frac{dx}{x\sqrt{4x^2-1}}$

6) $\int \frac{2dx}{\sqrt{x}(1+x)}$

7) $\int \frac{dx}{1+3x^2}$

8) $\int \frac{2\cos x}{1+\sin^2 x}dx$

9) $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$

10) $\int \frac{\tan^{-1} x}{1+x^2}dx$

Solution:

1) $\frac{1}{3}\int \frac{1}{\sqrt{1-(x^3)^2}}(3x^2\,dx) = \frac{1}{3}\sin^{-1} x^3 + c$

2) $\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3} + c$

3) $\frac{1}{2}\int \frac{2x}{1+(x^2)^2}dx = \frac{1}{2}\tan^{-1} x^2 + c$

4) $\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx = \sin^{-1}(\tan x) + c$

5) $\int \frac{2\,dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}(2x) + c$

$$6)\ \int \frac{2}{\sqrt{x}(1+x)}dx = 4\int \frac{\frac{1}{2\sqrt{x}}\,dx}{1+(\sqrt{x})^2} = 4\tan^{-1}\sqrt{x}+c$$

$$7)\ \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\,dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x)+c$$

$$8)\ 2\int \frac{\cos x\,dx}{1+(\sin x)^2} = 2\tan^{-1}(\sin x)+c$$

$$9)\ \int e^{\sin^{-1}x}\cdot\frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1}x}+c$$

$$10)\ \int \tan^{-1}x\cdot\frac{dx}{1+x^2} = \frac{(\tan^{-1}x)^2}{2}+c$$

4. Integrals of hyperbolic functions:

The integration formulas for the hyperbolic functions are:

$$19)\ \int \sinh u\cdot du = \cosh u + c$$

$$20)\ \int \cosh u\cdot du = \sinh u + c$$

$$21)\ \int \tanh u\cdot du = \ln(\cosh u) + c$$

$$22)\ \int \coth u\cdot du = \ln(\sinh u) + c$$

$$23)\ \int \sec h^2 u\cdot du = \tanh u + c$$

$$24)\ \int \csc h^2 u\cdot du = \coth u + c$$

$$25)\ \int \sec hu\cdot \tanh u\cdot du = -\sec hu + c$$

$$26)\ \int \csc hu\cdot \coth u\cdot du = -\csc hu + c$$

Example 4: **Evaluate the following integrals:**

1) $\int \frac{\cosh(\ln x)}{x} dx$

2) $\int \sinh(2x+1)dx$

3) $\int \frac{\sinh x}{\cosh^4 x} dx$

4) $\int x \cdot \cosh(3x^2)dx$

5) $\int \sinh^4 x \cdot \cosh x \; dx$

6) $\int \operatorname{sech}^2(2x-3)dx$

7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

8) $\int (e^{ax} - e^{-ax})dx$

9) $\int \frac{\sinh x}{1+\cosh x} dx$

10) $\int \operatorname{csch}^2 x \cdot \coth x \, dx$

Solution:

1) $\int \cosh(\ln x) \cdot \left(\frac{dx}{x}\right) = \sinh(\ln x) + c$

2) $\frac{1}{2}\int \sinh(2x+1) \cdot (2\,dx) = \frac{1}{2}\cosh(2x+1) + c$

3) $\int \frac{1}{\cosh^3 x} \cdot \frac{\sinh x}{\cosh x} \; dx = \int \operatorname{sech}^3 x \cdot \tanh x \; dx$

$= -\int \operatorname{sech}^2 x \cdot (-\operatorname{sech} x \cdot \tanh x \; dx) = -\frac{\operatorname{sech}^3 x}{3} + c$

4) $\frac{1}{6}\int \cosh(3x^2) \cdot (6x\,dx) = \frac{1}{6}\sinh(3x^2) + c$

5) $\int \sinh^4 x \cdot (\cosh x \; dx) = \frac{\sinh^5 x}{5} + c$

6) $\frac{1}{2}\int \operatorname{sech}^2(2x-3) \cdot (2 \; dx) = \frac{1}{2}\tanh(2x-3) + c$

7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \tanh x \; dx = \ln(\cosh x) + c$

8) $2\int \frac{e^{ax} - e^{-ax}}{2} dx = \frac{2}{a}\int \sinh ax \; (a\,dx) = \frac{2}{a}\cosh ax + c$

$$9)\ \int \frac{\sinh x\ dx}{1+\cosh x} = \ln(1+\cosh x)+c$$

$$10)\ -\int \operatorname{csc}hx\cdot(-\operatorname{csc}hx\cdot\coth x\ dx) = -\frac{\operatorname{csc}h^2 x}{2}+c$$

5. Integrals of inverse hyperbolic functions:

The integration formulas for the inverse hyperbolic functions are:

$$27)\ \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$28)\ \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$29)\ \int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c & \text{if } |u|<1 \\ \coth^{-1} u + c & \text{if } |u|>1 \end{cases} = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + c$$

$$30)\ \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sec}h^{-1}|u| + c = -\cosh^{-1}\left(\frac{1}{|u|}\right)+c$$

$$31)\ \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csc}h^{-1}|u| + c = -\sinh^{-1}\left(\frac{1}{|u|}\right)+c$$

Example 5: Evaluate the following integrals

$1)\ \int \frac{dx}{\sqrt{1+4x^2}}$ $\quad 2)\ \int \frac{dx}{\sqrt{4+x^2}}$ $\quad 3)\ \int \frac{dx}{1-x^2}$

$4)\ \int \frac{dx}{x\sqrt{4+x^2}}$ $\quad 5)\ \int \frac{\sec^2\theta\ d\theta}{\sqrt{\tan^2\theta-1}}$ $\quad 6)\ \int \tanh^{-1}(\ln\sqrt{x})\cdot\frac{dx}{x(1-\ln^2\sqrt{x})}$

Solution:

$$1)\ \frac{1}{2}\int \frac{2\ dx}{\sqrt{1+4x^2}} = \frac{1}{2}\sinh^{-1} 2x + c$$

$$2)\ \int \frac{\frac{1}{2}\ dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1}\frac{x}{2} + c$$

$$3)\ \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \qquad if \quad |x| < 1$$

$$= \coth^{-1} x + c \qquad if \quad |x| > 1$$

$$4)\ \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2}\int \frac{\frac{1}{2}\ dx}{\frac{x}{2}\sqrt{1+\left(\frac{x}{2}\right)^2}} = -\frac{1}{2}\operatorname{csc} h^{-1}\left|\frac{x}{2}\right| + c$$

$$5)\ \int \frac{1}{\sqrt{\tan^2\theta - 1}}\left(\sec^2\theta\ d\theta\right) = \cosh^{-1}(\tan\theta) + c$$

$$6)\quad let \quad u = \ln\sqrt{x} = \frac{1}{2}\ln x \qquad du = \frac{1}{2x}dx$$

$$\int \tanh^{-1}(\ln\sqrt{x})\cdot\frac{dx}{x(1-\ln^2\sqrt{x})} = \int \tanh^{-1} u\cdot\frac{2\,du}{1-u^2}$$

$$= 2\frac{\left(\tanh^{-1} u\right)^2}{2} + c = \left[\tanh^{-1}(\ln\sqrt{x})\right]^2 + c$$