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المحاضرة السادسة

MATRICES AND DETERMINANTS

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Matrices and Determinants

A matrix is a rectangular array of elements (scalars) from a field. The order, or size, of a matrix is specified by the number of rows and the number of columns, i.e. A an "m by n" matrix has m rows and n columns, and the element in the i *th* row and j *th* column is often denoted by a_{ij} :

$$A = \left[a_{ij}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

A vector is a matrix with a single row (or column) of n elements , i.e. the column vector is:-

$$A = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix} \quad \textit{and row vector is} \quad A = \left[a_1 \quad a_2 \quad . \quad . \quad a_n\right]$$

The matrix is square if the number of rows and columns are equal (i.e. $m = n$) and the elements a_{ij} of a square matrix are called the main diagonal.

The identity matrix: $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ **is square matrix with one in each main diagonal position and zeros else.**

The diagonal matrix $D = \begin{bmatrix} a_1 & 0 & .... & 0 \\ 0 & a_2 & .... & 0 \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ 0 & 0 & .... & a_n \end{bmatrix}$ has the elements $a_1, a_2, \ldots, a_n$ in its main diagonal position and zeros in all other locations, some of the a_i may be zero but not all.

A $n \times n$ triangular matrix has the pattern:-

$$\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ 0 & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ 0 & 0 & .... & a_{nn} \end{bmatrix} \quad or \quad \begin{bmatrix} a_{11} & 0 & .... & 0 \\ a_{21} & a_{22} & .... & 0 \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ a_{n1} & a_{n2} & .... & a_{nn} \end{bmatrix}$$

lower triangular matrix *upper triangular matrix*

The $m \times n$ null matrix:- $0 = \begin{bmatrix} 0 & 0 & .... & 0 \\ 0 & 0 & .... & 0 \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ 0 & 0 & .... & 0 \end{bmatrix}$ has zero in each of its positions.

Elementary operations with matrices and vectors

1. Equality:- Two $m \times n$ matrices and A and B are said to be equal if: $a_{ij} = b_{ij} \quad \forall$ *pairs of i and j.*

EX-1 – Find the values of x, y for the following matrix equation:

$$\begin{bmatrix} x-2y & 0 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -2 & x+y \end{bmatrix}$$

Sol. –

$$\left.\begin{array}{ll} x-2y=3 & ....(1) \\ x+y=6 & ....(2)*2 \end{array}\right\} \Rightarrow \begin{array}{c} x-2y=3 \\ \underline{2x+2y=12} \\ 3x=15 \Rightarrow \boxed{x=5} \end{array}$$

$$substitution\ \ x=5\ \ in(2)\ \Rightarrow\ 5+y=6\ \Rightarrow\ \boxed{y=1}$$

2. **Addition:-** **The sum of two matrices of like dimensions is the matrix of the sum of the corresponding elements. If:-**

$$A=\begin{bmatrix} a_{11} & a_{12} & .... & a_{1n} \\ a_{21} & a_{22} & .... & a_{2n} \\ .... & .... & .... & .... \\ a_{m1} & a_{m2} & .... & a_{mn} \end{bmatrix} \quad , \quad B=\begin{bmatrix} b_{11} & b_{12} & .... & b_{1n} \\ b_{21} & b_{22} & .... & b_{2n} \\ .... & .... & .... & .... \\ b_{m1} & b_{m2} & .... & b_{mn} \end{bmatrix}$$

then

$$A \mp B=\begin{bmatrix} a_{11}\mp b_{11} & a_{12}\mp b_{12} & .... & a_{1n}\mp b_{1n} \\ a_{21}\mp b_{21} & a_{22}\mp b_{22} & .... & a_{2n}\mp b_{2n} \\ .... & .... & .... & .... \\ a_{m1}\mp b_{m1} & a_{m2}\mp b_{m2} & .... & a_{mn}\mp b_{mn} \end{bmatrix}$$

thus:

1) *A+B = B+A*
2) *A+(B+C) = (A+B)+C*
3) *A-(B-C) = A-B+C*

***EX-2*- Find *A+B* and *A-B* if:-**

$$A=\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & -2 \end{bmatrix} \quad , \quad B=\begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

***Sol.*-**

$$A+B=\begin{bmatrix} 2+1 & 1-2 & 3+2 \\ 1+2 & 0+3 & -2-1 \end{bmatrix}=\begin{bmatrix} 3 & -1 & 5 \\ 3 & 3 & -3 \end{bmatrix}$$

$$A-B=\begin{bmatrix} 2-1 & 1-(-2) & 3-2 \\ 1-2 & 0-(+3) & -2-(-1) \end{bmatrix}=\begin{bmatrix} 1 & 3 & 1 \\ -1 & -3 & -1 \end{bmatrix}$$

3. <u>Multiplication by a scalar:-</u> The matrix A is multiplied by the scalar C by multiplying each element of A by c:-

$$CA = \begin{bmatrix} ca_{11} & ca_{12} & .... & ca_{1n} \\ ca_{21} & ca_{22} & .... & ca_{2n} \\ .... & .... & .... & .... \\ .... & .... & .... & .... \\ ca_{m1} & ca_{m2} & .... & ca_{mn} \end{bmatrix}$$

EX-3- Assume $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & -1 \end{bmatrix}$, find $3A$.

Sol.-

$$3A = \begin{bmatrix} 3*3 & 3*2 & 3*1 \\ 3*0 & 3*5 & 3*(-1) \end{bmatrix} = \begin{bmatrix} 9 & 6 & 3 \\ 0 & 15 & -3 \end{bmatrix}$$

4. <u>Matrix multiplication:-</u> For the matrix product AB to be defined it is necessary that the number of columns of A be equal to the number of rows of B. The dimensions of such matrices are said to be conformable. If A is of dimensions $m \times p$ and B is $p \times n$, then the ij th element of the product $C=AB$ is computed as:-

$$C_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}$$

This is the sum of the products of corresponding elements in the i th row of A and j th column of B. The dimensions of AB are of course $m \times n$.

EX-4- Assume $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 6 & 5 & 4 \\ -1 & 1 & -1 \\ 0 & 2 & 0 \end{bmatrix}$ find AB.

Sol.-

$$AB = \begin{bmatrix} 1*6+2(-1)+3*0 & 1*5+2*1+3*2 & 1*4+2(-1)+3*0 \\ -1*6+0(-1)+1*0 & -1*5+0*1+1*2 & -1*4+0(-1)+1*0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 13 & 2 \\ -6 & -3 & -4 \end{bmatrix}$$

Properties of multiplication:-

$a)\ A(B+C)=AB+AC$ *distributive law*

$b)\ A(BC)=(AB)C$ *associative law*

$c)\ AB\neq BA$ *commutative law does not hold*

$d)\ AI=IA=A$

EX-5- **Assume** $A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}$ *and* $B=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$ **, verify that** $AB\neq BA$ **.**

Sol.-

$$AB=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}7 & 1\\ 6 & 3\end{bmatrix} \quad \& \quad BA=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}=\begin{bmatrix}3 & 3\\ 2 & 7\end{bmatrix}$$

Hence $AB\neq BA$

5. Transpose of matrix:- Let A **is any** $m\times n$ **matrix the transpose of** A **is** $n\times m$ **matrix** A' **formed by interchanging the role of rows and columns.**

$$A'=\begin{bmatrix}a_{11} & a_{12} & .... & a_{1n}\\ a_{21} & a_{22} & .... & a_{2n}\\ .... & .... & .... & ....\\ a_{m1} & a_{m2} & .... & a_{mn}\end{bmatrix}'=\begin{bmatrix}a_{11} & a_{21} & .... & a_{m1}\\ a_{12} & a_{22} & .... & a_{m2}\\ .... & .... & .... & ....\\ a_{1n} & a_{2n} & .... & a_{mn}\end{bmatrix}$$

If a matrix is square and equal to its transpose, it is said to be symmetric, then $a_{ij}=a_{ji}$ **for all pairs of** i **and** j**.**

Properties of transpose are:-

$a)\ (A\mp B)'=A'\mp B'$

$b)\ (AB)'=B'A'$

$c)\ (A')'=A$

EX-6- **Assume** $A=\begin{bmatrix}3 & 2 & 5\\ 2 & -1 & 4\\ 5 & 4 & 0\end{bmatrix}$ *and* $B=\begin{bmatrix}4 & -1 & 0\\ 5 & 4 & 3\\ 2 & 1 & -1\end{bmatrix}$ **, show that:-**

$1)\ A$ *is symmetric matrix* $\qquad 2)\ (A+B)'=A'+B'$

$3)\ (AB)'=B'A'$

Sol.-

$$1)\ A' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix}' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} = A \Rightarrow A \text{ is a symmetric matrix.}$$

$$2)\ L.H.S. = (A+B)' = \begin{bmatrix} 7 & 1 & 5 \\ 7 & 3 & 7 \\ 7 & 5 & -1 \end{bmatrix}' = \begin{bmatrix} 7 & 7 & 7 \\ 1 & 3 & 5 \\ 5 & 7 & -1 \end{bmatrix}$$

$$R.H.S. = A' + B' = \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 2 \\ -1 & 4 & 1 \\ 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 1 & 3 & 5 \\ 5 & 7 & -1 \end{bmatrix} = L.H.S.$$

$$\therefore (A+B)' = A' + B'$$

$$3)\ L.H.S. = (AB)' = \begin{bmatrix} 32 & 10 & 1 \\ 11 & -2 & -7 \\ 40 & 11 & 12 \end{bmatrix}' = \begin{bmatrix} 32 & 11 & 40 \\ 10 & -2 & 11 \\ 1 & -7 & 12 \end{bmatrix}$$

$$R.H.S. = B'A' = \begin{bmatrix} 4 & 5 & 2 \\ -1 & 4 & 1 \\ 0 & 3 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 5 \\ 2 & -1 & 4 \\ 5 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 32 & 11 & 40 \\ 10 & -2 & 11 \\ 1 & -7 & 12 \end{bmatrix} = L.H.S.$$

$$\therefore (AB)' = B'A'$$

6. <u>Vector inner product:-</u> The inner product of two vectors with the same number of elements is defined to be the sum of the products of the corresponding elements:-

$$A'B = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_i\, b_i$$

Since the inner product is a scalar, hence $A'B = B'A$. Moreover, the inner product of two vectors may be taken the following term:-

$$A\,B' = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_n \\ a_2b_1 & a_2b_2 & \cdots & a_2b_n \\ \cdots & \cdots & \cdots & \cdots \\ a_nb_1 & a_nb_2 & \cdots & a_nb_n \end{bmatrix}$$

Which is $n \times n$ matrix.

<u>EX-7</u>- Let $A = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$**, find** $A'B$ *and* AB'

<u>Sol.</u>-

$$A'B = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 5 * 2 + (-2) * (-1) + 1 * 3 = 15$$

$$AB' = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 15 \\ -4 & 2 & -6 \\ 2 & -1 & 3 \end{bmatrix}$$

Determinants

The minor of the element a_{ij} in a matrix A is the determinant of the matrix that remains when the row and column containing a_{ij} are deleted. For example, let:-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ then the minor of } a_{21} \text{ is } \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \text{ then the minor of } a_{34} \text{ is } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$$

and so on.

The cofactor of a_{ij} is the determinant A_{ij} that is $(-1)^{i+j}$ times the minor of a_{ij} . Thus:-

$$for\ matrix\ (3\times 3)\Rightarrow A_{23}=(-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}=-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$for\ matrix\ (4\times 4)\Rightarrow A_{31}=(-1)^{3+1}\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}=\begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

With each square matrix A we associate a number $det\,A$ or $|A|$ or $|a_{ij}|$ called the determinant of A, calculated from the entries of A in the following way:-

$$for\ \ n=1\ ,\ A=[a]\Rightarrow |A|=a$$

$$for\ \ n=2\ ,\ A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\Rightarrow |A|=a_{11}\,a_{22}-a_{12}\,a_{21}$$

$$for\ \ n=3\ ,\ A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\Rightarrow |A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$|A|=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-\left(a_{13}a_{22}a_{31}+a_{11}a_{23}a_{32}+a_{12}a_{21}a_{33}\right)$$

The determinant of a square matrix can be calculated from the cofactors of any row or any column.

***<u>EX-8</u>*- Find the determinant of the matrix:-** $A=\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$

<u>Sol.-</u>

<u>1st method</u>

$$A=\begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & -2 \\ 2 & 3 & 1 \end{bmatrix}\begin{matrix} 2 & 1 \\ 3 & -1 \\ 2 & 3 \end{matrix}$$

$$=2(-1)\cdot 1+1(-2)\cdot 2+3\cdot 3\cdot 3-\left(3(-1)\cdot 2+2(-2)\cdot 3+1\cdot 3\cdot 1\right)$$
$$=36$$

2nd method

If we were to expand the determinant by cofactors according to elements of its third column, say, we would get:-

$$A = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$

$$= 3(-1)^{1+3}\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} + (-2)(-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 1(-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$$

$$= 3(9-(-2)) + 2(6-2) + (-2-3) = 36$$

Useful facts about determinants:-

F-1: If two rows of matrix are identical, the determinant is zero.

EX-9 **Show that:-** $\begin{vmatrix} 3 & -1 & 2 \\ 2 & -3 & 5 \\ 3 & -1 & 2 \end{vmatrix} = 0$

Sol.-

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & -3 & 5 \\ 3 & -1 & 2 \end{vmatrix}\begin{matrix} 3 & -1 \\ 2 & -3 \\ 3 & -1 \end{matrix} \quad = -18-15-4-(-18-15-4) = 0$$

F-2: Interchanging two rows of matrix changes the sign of its determinants.

EX-10 **Show that:-** $\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}\begin{matrix} 2 & 1 \\ -1 & 0 \\ 1 & -2 \end{matrix} \quad = 0+3+10-(0-12-4) = 29$$

$$R.H.S. = \begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}\begin{matrix} -1 & 0 \\ 2 & 1 \\ 1 & -2 \end{matrix} \quad = -(-4+0-12-(3+10+0)) = 29 = L.H.S.$$

$$\therefore \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = -\begin{vmatrix} -1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & -2 & 4 \end{vmatrix}$$

F-3: The determinant of the transpose of a matrix is equal to the original determinant.

EX-11 **Show that:-** $\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = 29 \quad from\ ex\text{-}10$$

$$R.H.S. = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix} \begin{matrix} 2 & -1 \\ 1 & 0 \\ 5 & 3 \end{matrix} = 0 + 10 + 3 - (0 - 12 - 4) = 29 \quad = \quad L.H.S.$$

$$\therefore \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 0 & -2 \\ 5 & 3 & 4 \end{vmatrix}$$

F-4: If each element of same row (or column) of a matrix is multiplied by a constant *C*, the determinant is multiplied by *C*.

EX-12 **Show that:-** $\begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}$

Sol.-

$$L.H.S. = \begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} \begin{matrix} 6 & 3 \\ -1 & 0 \\ 1 & -2 \end{matrix} = 0 + 9 + 30 - (0 - 36 - 12) = 87$$

$$R.H.S. = 3 * 29 = 87 = L.H.S.$$

$$\therefore \begin{vmatrix} 6 & 3 & 15 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{vmatrix}$$

F-5: If all elements of a matrix above the main diagonal (or all below it) are zero, the determinant of the matrix is the product of the elements on the main diagonal.

EX-13 **Find:-** $\begin{vmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 4 \end{vmatrix}$

Sol.-

$$\begin{vmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -1 & 4 \end{vmatrix}\begin{matrix} 5 & 0 \\ 2 & 3 \\ 1 & -1 \end{matrix} \quad = 60 + 0 + 0 - (0 - 0 - 0) = 60$$

$$Or\ directly \quad 5 * 3 * 4 = 60$$

F-6: If each element of a row of a matrix is multiplied by a constant C and the results added to a different row, the determinant is not changed.

EX-14 **Show that** $|A| = |B| \quad if \quad A = \begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 1 & -2 & 4 \end{bmatrix}$ **and** B **is the matrix resultant from multiplying row (*1*) by** 2 **and adding to row (*3*).**

i.e. $B = \begin{bmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 5 & 0 & 14 \end{bmatrix}$

Sol.-

$$|A| = 29 \quad from\ ex\text{-}10$$

$$|B| = \begin{vmatrix} 2 & 1 & 5 \\ -1 & 0 & 3 \\ 5 & 0 & 14 \end{vmatrix}\begin{matrix} 2 & 1 \\ -1 & 0 \\ 5 & 0 \end{matrix} \quad = 0 + 15 + 0 - (0 - 0 - 14) = 29$$

$$\therefore\ |A| = |B|$$

EX-15 **Find** $\begin{vmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & 0 & 2 \\ -1 & 2 & 1 & -2 \\ 0 & 1 & 2 & 1 \end{vmatrix}$

Sol.-

$$\overset{-2R_1+R_2}{\Rightarrow\Rightarrow\Rightarrow}\begin{vmatrix}1 & -2 & 3 & 1\\ 2 & 1 & 0 & 2\\ -1 & 2 & 1 & -2\\ 0 & 1 & 2 & 1\end{vmatrix}$$

Expanding the determinant by using the first column. $\Rightarrow\Rightarrow\Rightarrow 1\cdot\begin{vmatrix}5 & 2 & 0\\ 0 & 4 & -1\\ 1 & -6 & 1\end{vmatrix}\begin{matrix}5 & 2\\ 0 & 4\\ 1 & -6\end{matrix} = 20+6+0-(0-10+0)=36$

Linear Equations

There are many methods to solve a system of linear equations:
$AX=B$

I) <u>Row Reduction method</u> It is often possible to transform the linear equations step by step into an equivalent system of equations that is so simple it can be solved by inspection.

We start with $n\times(n+1)$ matrix $[A\vdots B]$ whose first n columns are the columns of A and whose last column is B. We are going to transform this augmented matrix with a sequence of elementary row operations into $[I\vdots S]$ where S is the solution of X.

***<u>EX-16</u>* Solve the following linear equation:**

$$\begin{aligned}2x+3y-4z&=-3\\ x+2y+3z&=3\\ 3x-y-z&=6\end{aligned}$$

<u>Sol.</u>

$$AX=B \quad where \quad A=\begin{bmatrix}2 & 3 & -4\\ 1 & 2 & 3\\ 3 & -1 & -1\end{bmatrix},\quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix},\quad B=\begin{bmatrix}-3\\ 3\\ 6\end{bmatrix}$$

$$[A\vdots B]=\begin{bmatrix}2 & 3 & -4 & \vdots -3\\ 1 & 2 & 3 & \vdots\ 3\\ 3 & -1 & -1 & \vdots\ 6\end{bmatrix}\underset{-3R_2+R_3}{\overset{-2R_2+R_1}{\Rightarrow\Rightarrow\Rightarrow}}\begin{bmatrix}0 & -1 & -10 & \vdots -9\\ 1 & 2 & 3 & \vdots\ 3\\ 0 & -7 & -10 & \vdots -3\end{bmatrix}$$

$$\underset{R_1\ and\ R_2}{\overset{inter\ change}{\Rightarrow\Rightarrow\Rightarrow}}\begin{bmatrix}1 & 2 & 3 & \vdots\ 3\\ 0 & -1 & -10 & \vdots -9\\ 0 & -7 & -10 & \vdots -3\end{bmatrix}\underset{-7R_2+R_3}{\overset{2R_2+R_1}{\Rightarrow\Rightarrow\Rightarrow}}\begin{bmatrix}1 & 0 & -17 & \vdots -15\\ 0 & -1 & -10 & \vdots -9\\ 0 & 0 & 60 & \vdots\ 60\end{bmatrix}$$

$$\underset{\frac{1}{6}R_3+R_2}{\overset{\frac{17}{60}R_3+R_1}{\Rightarrow\Rightarrow\Rightarrow}}\begin{bmatrix}1 & 0 & 0 & \vdots 2\\ 0 & -1 & 0 & \vdots\ 1\\ 0 & 0 & 60 & \vdots 60\end{bmatrix}\underset{R_3*\frac{1}{60}}{\overset{R_2*(-1)}{\Rightarrow\Rightarrow\Rightarrow}}\begin{bmatrix}1 & 0 & 0 & \vdots 2\\ 0 & 1 & 0 & \vdots -1\\ 0 & 0 & 1 & \vdots\ 1\end{bmatrix}=[I\vdots S]$$

$$Hence \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{x=2 \;\;,\; y=-1 \;\;,\; z=1}$$

II) <u>Cramer's Rule</u> When the determinant of the coefficient matrix *A* of the system *AX=B* is not zero (i.e. $|A| \neq 0$) the system has a unique solution that it may be found from the formulas:

$$X_i = \frac{|A_i|}{|A|}$$

Where $|A_i|$ is the determinant of the matrix, comes from replacing the *i th* column in *A* by the column of constant *B*.

<u>EX-17</u> **Resolve example 16 using Cramer's rule:**

<u>Sol.</u>

$$|A| = \begin{vmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{vmatrix}\begin{matrix} 2 & 3 \\ 1 & 2 \\ 3 & -1 \end{matrix} = -4+27+4-(-24-6-3) = 60$$

$$|A_1| = \begin{vmatrix} -3 & 3 & -4 \\ 3 & 2 & 3 \\ 6 & -1 & -1 \end{vmatrix}\begin{matrix} -3 & 3 \\ 3 & 2 \\ 6 & -1 \end{matrix} = 6+54+12-(-48+9-9) = 120$$

$$|A_2| = \begin{vmatrix} 2 & -3 & -4 \\ 1 & 3 & 3 \\ 3 & 6 & -1 \end{vmatrix}\begin{matrix} 2 & -3 \\ 1 & 3 \\ 3 & 6 \end{matrix} = -6-27-24-(-36+36+3) = -60$$

$$|A_3| = \begin{vmatrix} 2 & 3 & -3 \\ 1 & 2 & 3 \\ 3 & -1 & 6 \end{vmatrix}\begin{matrix} 2 & 3 \\ 1 & 2 \\ 3 & -1 \end{matrix} = 24+27+3-(-18-6+18) = 60$$

$$\therefore \quad x = \frac{|A_1|}{|A|} = \frac{120}{60} \Rightarrow x = 2$$

$$y = \frac{|A_2|}{|A|} = \frac{-60}{60} \Rightarrow y = -1$$

$$z = \frac{|A_3|}{|A|} = \frac{60}{60} \Rightarrow z = 1$$

The same result in ex - 16.

Problems – 8

1) Let $A=\begin{bmatrix}1&2\\0&4\end{bmatrix}$, $B=\begin{bmatrix}1&2&3\\-1&4&-2\end{bmatrix}$, $C=\begin{bmatrix}3&1\\4&-1\\0&2\end{bmatrix}$,

$D=\begin{bmatrix}1&0&4\\0&1&2\\0&-1&1\end{bmatrix}$, $E=\begin{bmatrix}3&-1\\4&2\end{bmatrix}$. **Find:-**

a) AB *b) DC* *c) (D+I)C* *d) DC+C* *e) DCB*
f) EI *g) 3A+E* *h) -5E+A* *i) E(2B)*

$$\left(ans.:\ a)\begin{bmatrix}-1&10&-1\\-4&16&-8\end{bmatrix}\ b)\begin{bmatrix}3&9\\4&3\\-4&3\end{bmatrix}\ c,d)\begin{bmatrix}6&10\\8&2\\-4&5\end{bmatrix}\ e)\begin{bmatrix}-6&42&-9\\1&20&6\\-7&4&-18\end{bmatrix}\right.$$
$$\left.f)\begin{bmatrix}3&-1\\4&2\end{bmatrix}\ g)\begin{bmatrix}6&5\\4&14\end{bmatrix}\ h)\begin{bmatrix}-14&7\\-20&-6\end{bmatrix}\ i)\begin{bmatrix}8&4&22\\4&32&16\end{bmatrix}\right)$$

2) Find the value of x **:-**

$$\begin{bmatrix}x&4&1\end{bmatrix}\begin{bmatrix}2&1&0\\1&0&2\\0&2&4\end{bmatrix}\begin{bmatrix}x\\-7\\5/4\end{bmatrix}=0$$

$$\left(ans.:\ x=\frac{1}{2}\quad or\quad x=1\right)$$

3) Find v **and** w **if:** $\begin{bmatrix}5&w\end{bmatrix}=v\begin{bmatrix}-2&1\end{bmatrix}$.

$$\left(ans.:\ w=v=-\frac{5}{2}\right)$$

4) Let $A=\begin{bmatrix}1&-1&2\\0&1&4\end{bmatrix}$, $B=\begin{bmatrix}0&2\\-1&3\\5&-2\end{bmatrix}$, **Find:-**

$a)\ 2A+B'$ $b)\ B'A'-I$

$$\left(ans.:a)\begin{bmatrix}2&-3&9\\2&5&6\end{bmatrix}\ b)\begin{bmatrix}10&19\\-5&-6\end{bmatrix}\right)$$

5) Let $A=\begin{bmatrix}3&0&1\\0&1&2\\-1&1&5\end{bmatrix}$, $B=\begin{bmatrix}1&-2&0\\-1&3&2\end{bmatrix}$, **Find:-**

$(2A - I)B'$ **and show that** $(AB)' = B'A'$

$$\left(ans.: \begin{bmatrix} 5 & -1 \\ -2 & 11 \\ -6 & 26 \end{bmatrix} \right)$$

6) **For what value of** x **will:** $\begin{vmatrix} x & x & 1 \\ 2 & 0 & 5 \\ 6 & 7 & 1 \end{vmatrix} = 0$ **?**

$$(ans.: \ x = 2)$$

7) **Let** A **be an arbitrary 3 by 3 matrix and let** R_{12} **be the matrix obtained from the 3 by 3 identity matrix by interchanging row 1 and 2 :** $R_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$**. a) Compute** $R_{12}A$ **and show that you would get the same result by interchanging rows 1 and 2 of** A**. b) Compute** AR_{12} **and show that the result is that you would get by interchanging column 1 and 2 of** A**.**

$$\left(ans.: \ a) \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \ b) \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix} \right)$$

8) **Solve the following determinants:-**

$$a)\begin{vmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 3 \end{vmatrix} \quad b)\begin{vmatrix} 2 & -1 & -2 \\ -1 & 2 & 1 \\ 3 & 0 & -3 \end{vmatrix} \quad c)\begin{vmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$$d)\begin{vmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \\ 2 & 0 & 1 \end{vmatrix} \quad e)\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & -1 & 0 & 7 \\ 3 & 0 & 2 & 1 \end{vmatrix} \quad f)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 2 \end{vmatrix}$$

$$g)\begin{vmatrix} 1 & -1 & 2 & 3 \\ 2 & - & 2 & 6 \\ 1 & 0 & 2 & 3 \\ -2 & 2 & 0 & -5 \end{vmatrix} \quad h)\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix}$$

$$\left(\begin{array}{llll} ans.: \ a)-5 & b)\,0 & c)-7 & d)\,6 \\ \quad e)\,38 & f)\,1 & g)\,2 & h)-1 \end{array} \right)$$

9) Solve the following system of equations:-

a) $x + 8y = 4$
$3x - y = -13$

b) $2x + 3y = 5$
$3x - y = 2$

c) $x + y + z = 2$
$2x - y + z = 0$
$x + 2y - z = 4$

d) $2x + y - z = 2$
$x - y + z = 7$
$2x + 2y + z = 4$

e) $2x - 4y = 6$
$x + y + z = 1$
$5y + 7z = 10$

f) $x - z = 3$
$2y - 2z = 2$
$2x + z = 3$

g) $x_1 + x_2 - x_3 + x_4 = 2$
$x_1 - x_2 + x_3 + x_4 = -1$
$x_1 + x_2 + x_3 - x_4 = 2$
$x_1 + x_3 + x_4 = -1$

h) $2x - 3y + 4z = -19$
$6x + 4y - 2z = 8$
$x + 5y + 4z = 23$

(ans.: a) $x = -4,\ y = 1$ — b) $x = y = 1$

c) $x = \frac{6}{7},\ y = \frac{10}{7},\ z = -\frac{2}{7}$ — d) $x = 3,\ y = -2,\ z = 2$

e) $x = 0,\ y = -\frac{3}{2},\ z = \frac{5}{2}$ — f) $x = 2,\ y = 0,\ z = -1$

g) $x_1 = 2,\ x_2 = 0,\ x_3 = x_4 = -\frac{3}{2}$ — h) $x = -2,\ y = 5,\ z = 0$)