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Mathematics 1

Lecture 2

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الرياضيات الاساسية: المرحلة الاولى

مادة الرياضيات 1

المحاضرة الثانية

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Contents

1 Inverse of a Matrix

The inverse of a matrix A (denoted as A^{-1}) is a matrix such that:

$$A \times A^{-1} = A^{-1} \times A = I$$

where I is the identity matrix.

Conditions for Inverses:

1. A must be a square matrix $(n{\times}n)$.

2. $det(A) \neq 0$ (the determinant of A is non-zero).

Formula for 2×2 Matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

its inverse is:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Formula for 3×3 Matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

its inverse is:

$$A^{-1} = \frac{1}{\text{det}(A)} \text{adj}(A),$$

where adj(A) is the adjugate (transpose of the cofactor matrix) of **A**.

Example 1.1. Find inverse of matrix $A = \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix}$

Sol. **Step 1**

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

such that $ad - bc \neq 0$.

Step 2: Calculate the Determinant of A

For our matrix **A**:

$$\det(\mathbf{A}) = 1 \cdot 5 - (-2) \cdot 4 = 5 + 8 = 13.$$

Since the determinant is not zero, the inverse exists.

Step 3: Calculate the Inverse

$$\mathbf{A}^{-1} = \frac{1}{13} \begin{pmatrix} 5 & 2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} & \frac{2}{13} \\ -\frac{4}{13} & \frac{1}{13} \end{pmatrix}.$$

□

Example 1.2. Find inverse of matrix $\mathbf{A} = \begin{pmatrix} 3 & -6 & 7 \\ 4 & 0 & -5 \\ 5 & -8 & 6 \end{pmatrix}$

Sol. **Step 1**

$$\det(A) = 3 \cdot \begin{vmatrix} 0 & -5 \\ -8 & 6 \end{vmatrix} - (-6) \cdot \begin{vmatrix} 4 & -5 \\ 5 & 6 \end{vmatrix} + 7 \cdot \begin{vmatrix} 4 & 0 \\ 5 & -8 \end{vmatrix}$$

$$= 3(0 \cdot 6 - (-5) \cdot (-8)) + 6(4 \cdot 6 - (-5) \cdot 5) + 7(4 \cdot (-8) - 0 \cdot 5)$$

$$= 3(0 - 40) + 6(24 + 25) + 7(-32 - 0)$$

$$= 3(-40) + 6(49) + 7(-32)$$

$$= -120 + 294 - 224$$

$$= -50$$

To find the adjugate (adjoint) of matrix **A**, we need to find the cofactor matrix and then transpose it.

1.Calculate the minor matrices for each element:

- For $a_{11} = 3$:

$$M_{11} = (-1)^{1+1} \begin{vmatrix} 0 & -5 \\ -8 & 6 \end{vmatrix} = 0 \cdot 6 - (-5)(-8) = -40$$

- For $a_{12} = -6$:

$$M_{12} = (-1)^{1+2} \begin{vmatrix} 4 & -5 \\ 5 & 6 \end{vmatrix} = -(4 \cdot 6 - (-5) \cdot 5) = -49$$

- For $a_{13} = 7$:

$$M_{13}(-1)^{1+3} = \begin{vmatrix} 4 & 0 \\ 5 & -8 \end{vmatrix} = 4(-8) - 0 \cdot 5 = -32$$

- For $a_{21} = 4$:

$$M_{21} = (-1)^{2+1} \begin{vmatrix} -6 & 7 \\ -8 & 6 \end{vmatrix} = -((-6)(6) - 7(-8)) = -20$$

- For $a_{22} = 0$:

$$M_{22} = (-1)^{2+2} \begin{vmatrix} 3 & 7 \\ 5 & 6 \end{vmatrix} = 3 \cdot 6 - 7 \cdot 5 = -17$$

- For $a_{23} = -5$:

$$M_{23} = (-1)^{2+3} \begin{vmatrix} 3 & -6 \\ 5 & -8 \end{vmatrix} = -(3(-8) - (-6)5) = -6$$

- For $a_{31} = 5$:

$$M_{31} = (-1)^{3+1} \begin{vmatrix} -6 & 7 \\ 0 & -5 \end{vmatrix} = (-6)(-5) - 7 \cdot 0 = 30$$

- For $a_{32} = -8$:

$$M_{32} = (-1)^{3+2} \begin{vmatrix} 3 & 7 \\ 4 & -5 \end{vmatrix} = -(3(-5) - 7 \cdot 4) = 43$$

- For $a_{33} = 6$:

$$M_{33} = (-1)^{3+3} \begin{vmatrix} 3 & -6 \\ 4 & 0 \end{vmatrix} = 3 \cdot 0 - (-6)(4) = 24$$

2. Form the cofactor matrix:

$$\mathbf{C} = \begin{bmatrix} -40 & -49 & -32 \\ -20 & -17 & -6 \\ 30 & 43 & 24 \end{bmatrix}$$

3. Transpose the cofactor matrix to obtain the adjugate (adjoint) matrix**:

$$\operatorname{adj}(\mathbf{A}) = \mathbf{C}^\top = \begin{bmatrix} -40 & -20 & 30 \\ -49 & -17 & 43 \\ -32 & -6 & 24 \end{bmatrix}$$

5. Find the inverse by dividing the adjugate by the determinant:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathrm{adj}(\mathbf{A}) = \frac{1}{-50}\begin{bmatrix} -40 & -20 & 30 \\ -49 & -17 & 43 \\ -32 & -6 & 24 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{-40}{-50} & \frac{-20}{-50} & \frac{30}{-50} \\ \frac{-49}{-50} & \frac{-17}{-50} & \frac{43}{-50} \\ \frac{-32}{-50} & \frac{-6}{-50} & \frac{24}{-50} \end{bmatrix} = \begin{bmatrix} \frac{4}{5} & \frac{2}{5} & -\frac{3}{5} \\ \frac{49}{50} & \frac{17}{50} & -\frac{43}{50} \\ \frac{16}{25} & \frac{3}{25} & -\frac{12}{25} \end{bmatrix}$$

□

Homework of Inverse Matrix

1. Find A^{-1}, where

$$A = \begin{pmatrix} 2 & -7 \\ 1 & 5 \end{pmatrix}$$

2. Find A^{-1}, where

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 5 & 2 & -6 \\ 3 & -1 & -4 \end{pmatrix}$$

2 Solution of a System of Linear Equations

A system of linear equations is a set of equations where each equation is linear. Consider the following system of m equations with n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

where a_{ij} are the coefficients, x_j are the variables, and b_i are the constants. In matrix form, the system can be expressed as:

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

There are several methods to solve this system:

- **Direct Methods:** Gauss Elimination Method
- **Iterative Methods:** Gauss-Seidel
- **Solution by Cramer's Rule**
- **Solution by Matrix Inversion**

2.1 Cramer's Rule

Cramer's Rule is a mathematical theorem used to solve systems of linear equations with as many equations as unknowns, provided the determinant of the coefficient matrix is non-zero.

General Form

Consider a system of linear equations:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$

We can represent this system in matrix form as:

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

For Cramer's Rule to apply, we need $\det(\mathbf{A}) \neq 0$. We then solve for x, y, and z by computing determinants of matrices where each column of **A** is replaced by **b** one at a time:

1. Solution for x:

$$x = \frac{\det(\mathbf{A}_x)}{\det(\mathbf{A})}$$

where $\mathbf{A}_x$ is the matrix obtained by replacing the first column of $\mathbf{A}$ with $\mathbf{b}$:

$$\mathbf{A}_x = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}.$$

2. Solution for y:

$$y = \frac{\det(\mathbf{A}_y)}{\det(\mathbf{A})}$$

where $\mathbf{A}_y$ is the matrix obtained by replacing the second column of $\mathbf{A}$ with $\mathbf{b}$:

$$\mathbf{A}_y = \begin{pmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{pmatrix}.$$

3. Solution for z:

$$z = \frac{\det(\mathbf{A}_z)}{\det(\mathbf{A})}$$

where $\mathbf{A}_z$ is the matrix obtained by replacing the third column of $\mathbf{A}$ with $\mathbf{b}$:

$$\mathbf{A}_z = \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{pmatrix}.$$

Example 2.1. Solve the system of linear equations using Cramer's rule

$$\begin{cases} x - 2y = 3, \\ 4x + 5y = 12. \end{cases}$$

Sol. This system can be written in matrix form as:

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$$

The solution can be found using Cramer's rule:

$$x_i = \frac{\det(\mathbf{A}_i)}{\det(\mathbf{A})}$$

First, calculate the determinant of **A**:

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & -2 \\ 4 & 5 \end{vmatrix} = 1 \cdot 5 - (-2) \cdot 4 = 5 + 8 = 13$$

Next, calculate the determinants of $\mathbf{A}_x$ and $\mathbf{A}_y$:

$$\mathbf{A}_x = \begin{bmatrix} 3 & -2 \\ 12 & 5 \end{bmatrix} \Rightarrow \det(\mathbf{A}_x) = \begin{vmatrix} 3 & -2 \\ 12 & 5 \end{vmatrix} = 3 \cdot 5 - (-2) \cdot 12 = 15 + 24 = 39$$

$$\mathbf{A}_y = \begin{bmatrix} 1 & 3 \\ 4 & 12 \end{bmatrix} \Rightarrow \det(\mathbf{A}_y) = \begin{vmatrix} 1 & 3 \\ 4 & 12 \end{vmatrix} = 1 \cdot 12 - 3 \cdot 4 = 12 - 12 = 0$$

Therefore, the solutions are:

$$x = \frac{\det(\mathbf{A}_x)}{\det(\mathbf{A})} = \frac{39}{13} = 3$$

$$y = \frac{\det(\mathbf{A}_y)}{\det(\mathbf{A})} = \frac{0}{13} = 0$$

Thus, the solution to the system is:

$$x = 3, \quad y = 0$$

□

Homework of Cramer's Rule

1. Solve the system of linear equations using Cramer's rule

$$\begin{cases} 3x + 5y = 2, \\ -x + 2y = 0. \end{cases}$$

2. Solve the system of linear equations using Cramer's rule

$$\begin{cases} x + 7y = 2, \\ -x + 5y = 0. \end{cases}$$

2.2 Inverse Matrix Method

To solve a system of linear equations using the inverse matrix method, consider the system:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1, \\ a_{21}x + a_{22}y + a_{23}z = b_2, \\ a_{31}x + a_{32}y + a_{33}z = b_3. \end{cases}$$

This system can be represented in matrix form as:

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

To find **x** using the inverse of **A**, we need to have det(**A**) $\neq 0$. When the inverse of

$\mathbf{A}$, denoted $\mathbf{A}^{-1}$, exists, the solution can be found by:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

Steps for Solving Using the Inverse Matrix

1. Compute the inverse of $\mathbf{A}$:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\operatorname{adj}(\mathbf{A}),$$

where $\operatorname{adj}(\mathbf{A})$ is the adjugate (transpose of the cofactor matrix) of $\mathbf{A}$.

2. Multiply $\mathbf{A}^{-1}$ by $\mathbf{b}$ to find $\mathbf{x}$:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

Example 2.2. Solve the system of linear equations using Inverse Matrix Method

$$\begin{cases} x - 2y = 3, \\ 4x + 5y = 12. \end{cases}$$

Sol. To solve the following system of linear equations, we express it in matrix form $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 4 & 5 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}.$$

Step 1: Calculate the Determinant of A

For our matrix $\mathbf{A}$:

$$\det(\mathbf{A}) = 1 \cdot 5 - (-2) \cdot 4 = 5 + 8 = 13.$$

Since the determinant is not zero, the inverse exists.

Step 2: Calculate the Inverse

Using the formula for the inverse of a 2x2 matrix:

$$\mathbf{A}^{-1} = \frac{1}{13}\begin{pmatrix} 5 & 2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{13} & \frac{2}{13} \\ -\frac{4}{13} & \frac{1}{13} \end{pmatrix}.$$

Step 3: Solve for x

Now we can find **x** using:

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}.$$

Calculating this gives:

$$\mathbf{x} = \begin{pmatrix} \frac{5}{13} & \frac{2}{13} \\ -\frac{4}{13} & \frac{1}{13} \end{pmatrix}\begin{pmatrix} 3 \\ 12 \end{pmatrix}.$$

Performing the matrix multiplication:

1. First row:

$$\frac{5}{13}(3) + \frac{2}{13}(12) = \frac{15}{13} + \frac{24}{13} = \frac{39}{13} = 3.$$

2. Second row:

$$-\frac{4}{13}(3) + \frac{1}{13}(12) = -\frac{12}{13} + \frac{12}{13} = 0.$$

Thus, we have:

$$\mathbf{x} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Therefore, the solution to the system of equations is:

$$x = 3, \quad y = 0.$$

□