



جامعة المستقبل
AL MUSTAQBAL UNIVERSITY

كلية العلوم
قسم الكيمياء الحياتية

POLAR COORDINATES

Mathematics

المرحلة الاولى

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المحاضرة الثانية

REAL AND COMPLEX NUMBERS

Polar Coordinates

Let r and θ be polar coordinates of the point (x, y) corresponding to a nonzero complex number $z = x + iy$

Since, $x = r \cos \theta$, $y = r \sin \theta$

$$\boxed{z = r(\cos \theta + i \sin \theta)} , \quad \text{polar form of number } z$$

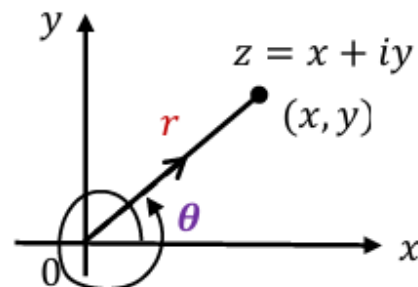
For example,

$$1 + i = \sqrt{2} \left[\cos \left(\frac{\pi}{4} \right) + i \sin \left(\frac{\pi}{4} \right) \right] = \sqrt{2} \left[\cos \left(\frac{-7\pi}{4} \right) + i \sin \left(\frac{-7\pi}{4} \right) \right]$$

The number θ is called an *argument* of z , (also called, 'phase' and 'angle')

θ is the directed angle measured from the positive x -axis to a point p and called (*argument* of z) and is denoted by : $\arg$; $\theta = \arg z_1$

Hence the angles will be measured in *radian* and positioned in the *counterclockwise sense*. $\boxed{\tan \theta = \frac{y}{x}}$



Important identity

1. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
2. $\arg(z_1 / z_2) = \arg(z_1) - \arg(z_2)$

Question : Find the value of (iz)

$$iz = \underbrace{(1) \left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right]}_i \underbrace{r(\cos \theta + i \sin \theta)}_z$$

$$iz = r \left[\cos \left(\theta + \frac{\pi}{2} \right) + i \sin \left(\theta + \frac{\pi}{2} \right) \right]$$

iz has an important applications in Optics i.e. (*circular polarization*).

Example:

$$\theta_1 = \arg(1 + i) \quad , \quad \theta_1 = \arg(-1 - i) \Rightarrow \tan \theta_1 = \tan \theta_2 = 1 \dots \text{PLOT!!}$$

Question : Use Cartesian coordinate system representation of z , simplified the multiplication of two different complex numbers.

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + y_2 x_1)$$

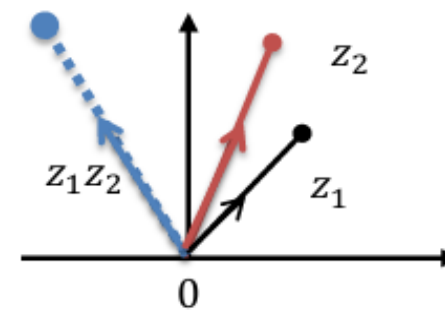
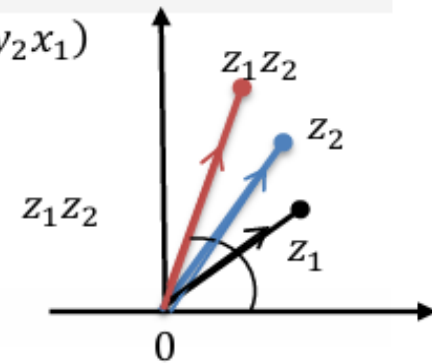
$$(1 + i)(2 + 3i) = 2 + 5i - 3 = -1 + 5i$$

$$z_1 = (1 + i) \rightarrow x_1 = 1, y_1 = 1, r_1 = \sqrt{2}, \theta_1 = 45^\circ,$$

$$\tan^{-1} \frac{3}{2} \text{ or } \theta_2 = 56.31^\circ,$$

For the *final results* $-1 + 5i$, will be :

$$r = 5.1, \theta = \tan^{-1} \frac{5}{-1}, \theta = 101.31^\circ$$



Exponential Function

الدالة الأسية العقدية

The polar form is $z = r(\cos \theta + i \sin \theta)$,

or , $z = re^{i\theta}$

Where $e^{i\theta} = (\cos \theta + i \sin \theta)$, is *Euler's formula* , and $e^{i\theta}$ (*complex exponential function*)

NOTE:

$$(1) z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$(2) \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Question : Relate trigonometric function with complex exponential function

الربط بين الدوال الاسية العقدية والدوال المثلثية الحقيقية

Proof:

$$\boxed{e^{-i\theta} = \cos \theta - i \sin \theta}$$

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \dots (1)$$

$$\underline{e^{-i\theta} = \cos \theta - i \sin \theta} \quad \dots (2)$$

Adding (1) and (2) : $e^{i\theta} + e^{-i\theta} = 2 \cos \theta \Rightarrow \boxed{\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}}$

subtracting (1) and (2)

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \dots (1)$$

$$\underline{e^{-i\theta} = \cos \theta - i \sin \theta} \quad \dots (2)$$

$$e^{i\theta} - e^{-i\theta} = 2i \sin \theta \rightarrow \boxed{\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2}}$$

Powers of complex numbers

$$\boxed{z^n = r^n e^{in\theta}}, \quad (n = 0, \pm 1, \pm 2, \dots) \quad , \text{ where } (e^{i\theta})^n = e^{in\theta}$$

Or $\boxed{(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta = \cos \phi + i \sin \phi},$
($n = 0, \pm 1, \pm 2, \dots$) (*de Moivre's theorem!!*)

Where $\boxed{\phi = n\theta}$

Roots of complex number z

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} \left[\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right], \quad \boxed{k = 0, 1, 2, \dots, n-1}$$

Example

1. Solve the equation $z^3 + 1 = 0$, or in other words
2. Find z for the equation $z = (-1)^{\frac{1}{3}}$
3. Find the third roots(**ONLY**) of the equation $z^3 + 1 = 0$

Solution

$$z^3 = -1, = (-1)^{\frac{1}{3}},$$

$$x = -1, y = 0, r = \sqrt{x^2 + y^2} \Rightarrow r = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \theta = \pi, \quad n = 3, \quad k = 0, 1, 2$$

$$\begin{aligned} \text{1}^{\text{st}} \text{ root } \boxed{k = 0}, \quad z_1 &= (1) \left[\cos \frac{\pi + 2(0)\pi}{3} + i \sin \frac{\pi + 2(0)\pi}{3} \right] \\ &= \frac{1}{2} + i0.866 = 0.5 + i0.866 \end{aligned}$$

$$\text{2}^{\text{nd}} \text{ root } \boxed{k = 1},$$

$$\begin{aligned} z_2 &= \left[\cos \frac{\pi + 2\pi}{3} + i \sin \frac{\pi + 2\pi}{3} \right] \\ &= \cos \pi + i \sin \pi = -1 + 0i \end{aligned}$$

$$\text{3}^{\text{rd}} \text{ root } \boxed{k = 2},$$

$$z_3 = \left[\cos \frac{\pi + 4\pi}{3} + i \sin \frac{\pi + 4\pi}{3} \right]$$

$$= \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = 0.5 - 0.866i$$

The 3rd root $k = 2$ was $(0.5, -0.866)$

Examples

1. Find one value of $\arg(z)$ when

$$\text{a) } z = \frac{-2}{1+i\sqrt{3}}, \quad \text{b) } z = \frac{i}{-2-2i}, \quad \text{c) } z = (\sqrt{3}-i)^6$$

Solution:

$$\text{a) } z = \frac{-2}{1+i\sqrt{3}}, \quad \frac{-2}{1+i\sqrt{3}} \cdot \frac{1-i\sqrt{3}}{1-i\sqrt{3}} = \frac{(-2)(1-\sqrt{3}i)}{1+3}$$

$$= \frac{-2+2\sqrt{3}i}{4} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$$

$$\therefore x = -\frac{1}{2} = -0.5$$

$$y = \frac{1}{2}\sqrt{3} = 0.866 \Rightarrow \theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1/2\sqrt{3}}{-0.5}\right)$$

$$\theta = \tan^{-1}(-\sqrt{3}) \Rightarrow \boxed{\theta = -60^\circ}, \theta = 180^\circ - 60^\circ = 120^\circ$$

$$\text{b) } \frac{i}{-2-2i},$$

$$\frac{i}{-2-2i} = \frac{i}{-2-2i} * \frac{-2+2i}{-2+2i} = \frac{i(-2+2i)}{8}$$

$$\frac{i}{-z-2i} = \frac{-2i+2}{8}$$

$$\frac{i}{-z-2i} = \frac{1-i}{4} = \frac{1}{4} - \frac{1}{4}i$$

$$x = \frac{1}{4}, y = -\frac{1}{4} \Rightarrow \theta = \tan^{-1} \left(\frac{-1/4}{1/4} \right) = \tan^{-1}(-1) = -45^\circ = 315^\circ = -\frac{\pi}{4} \equiv \frac{3\pi}{4}$$

$$\text{c) } z = (\sqrt{3} - i)^6$$

$$x = \sqrt{3}$$

$$y = -1, \quad \theta = \tan^{-1} \left(\frac{y}{x} \right) = \tan^{-1} \left(-\frac{1}{\sqrt{3}} \right) = -30^\circ$$

$$n\theta = 6 * \theta = -30^\circ * 6 = -180^\circ = 360^\circ - 180^\circ = \pi$$