Al-Mustaqbal University

Department (Biomedical Engineering)

Class (First Stage)

Subject (Physics)

Lecturer (Asst.lec.Hiba Diaa Alrubaie)

1st/term – Lect. (Units, Trigonometry and vectors.part 2)

Example 2:

Find real numbers x, y and z such that

$xi + 2yj - zk + 3i - j = 4i + 3k$

Solution:

Since $(x + 3)i + (2y - 1)j + (-z)k = 4i + 3k$

Comparing both sides, we get

$x + 3 = 4, \quad 2y - 1 = 0, \quad -z = 3$

$x = 1, \quad y = \frac{1}{2}, \quad z = -3$

Note 2: If $\overline{r_1} = x_1 i + y_1 j + z_1 k$

$r_2 = x_2 i + y_2 j + z_2 k$

Then the sum vector =

$$\overline{r_1} + \overline{r_2} = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$$

Example 3

$\vec{a} = 3i - 2j + 5k$ and $\vec{b} = -2i - j + k$.

Find $2\vec{a} - 3\vec{b}$ and also its unit vector.

Solution:

$$\begin{aligned} 2\vec{a} - 3\vec{b} &= 2(3i - 2j + 5k) - 3(-2i - j + k) \\ &= 6i - 4j + 10k + 6i + 3j - 3k \\ &= 12i - j + 7k \end{aligned}$$

If we denote $2\vec{a} - 3\vec{b} = \vec{c}$, then $\vec{c} = 12i - j + 7k$

and $|\vec{c}| = \sqrt{12^2 + (-1)^2 + 7^2} = \sqrt{144 + 1 + 49} = \sqrt{194}$

Therefore, $\hat{c} = \dfrac{\vec{c}}{|\vec{c}|} = \dfrac{12i - j + 7k}{\sqrt{194}}$

$$\hat{c} = \frac{12}{\sqrt{194}} i - \frac{1}{\sqrt{194}} j + \frac{1}{\sqrt{194}} k$$

Note 3: Two vectors $\vec{r_1} = x_1 i + y_1 j + z_1 k$ and $\vec{r_2} = x_2 i + y_2 j + z_2 k$ are parallel if and only if $\dfrac{x_1}{x_2} = \dfrac{y_1}{y_2} = \dfrac{z_1}{z_2}$.

Find the vector whose magnitude is 5 and is in the direction of $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$:

Given vector: $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Magnitude:

$$\|\mathbf{v}\| = \sqrt{4^2 + (-3)^2 + 1^2} = \sqrt{16 + 9 + 1} = \sqrt{26}.$$

Unit vector:

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{26}}(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

Vector of magnitude 5:

$$\mathbf{u} = 5 \cdot \hat{\mathbf{v}} = \frac{5}{\sqrt{26}}(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

• Product of Vectors:

Scalar product of two vectors in terms of their rectangular components.

For the two vectors

$$a = a_1 i + a_2 j + a_3 k$$

and $$b = b_1 i + b_2 j + b_3 k$$

the dot product is given as,

$$a \ . \ b = (a_1 i + a_2 j + a_3 k) \ . \ (b_1 i + b_2 j + b_3 k)$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3 \text{ as } i^2 = j^2 = k^2 = 1$$

$$\text{and } i \ . \ j = j \ . \ k = k \ . \ i = 0$$

Also a and b are perpendicular if and only if $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

Example :

If $a = 3i + 4j - k$, $b = -2i + 3j + k$ find $a \ . \ b$

Solution:

$$a \ . \ b = (3i + 4j - k) \ . \ (-2i + 3j + k)$$

$$= -6 + 12 - 1$$

$$= 5$$

cross product

If $\bar{a} = a_1 i + a_2 j + a_3 k$

and $\bar{b} = b_1 i + b_2 j + b_3 k$

then $\bar{a} \times \bar{b} = (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k)$

$= (a_1 b_2 k - a_1 b_3 j - a_2 b_1 k + a_2 b_3 j + a_3 b_1 j - a_3 b_2 j)$

$= (a_2 b_3 - a_3 b_2) i - (a_1 b_3 - a_3 b_1) j + (a_1 b_2 - a_2 b_1) k$

This result can be expressed in determinant form as

$$\overline{a} \times \overline{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example

Find the area of the parallelogram with adjacent sides,

a = i − j + k, and b = 2j − 3k

Solution:

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 2 & -3 \end{vmatrix}$$

$= i(3 - 2) - j(-3 - 0) + k(2 + 0)$

$= i + 3j + 2k$

Area of parallelogram = $| a \times b | = \sqrt{1 + 9 + 4}$

$= \sqrt{14}$ square unit.

Example :

Find the area of the triangle whose vertices are A(0, 0, 0), B(1, 1, 1) and C(0, 2, 3)

Solution:

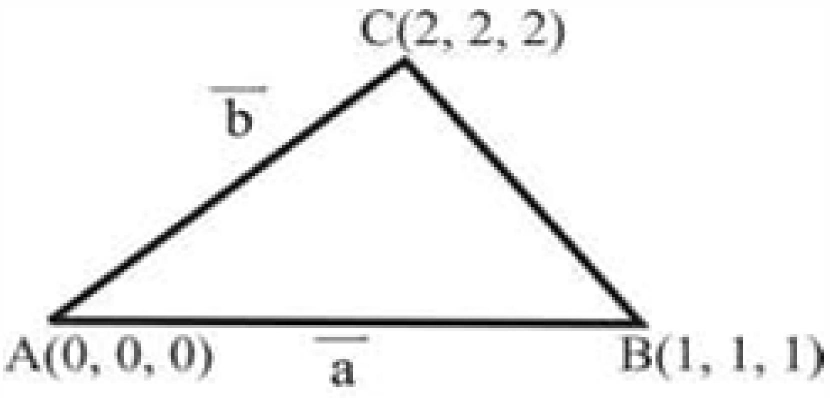

Fig. 21

Since AB $= (1 - 0, 1 - 0, 1 - 0)$
$= (1, 1, 1)$

and AC $= (0 - 0, 2 - 0, 3 - 0)$

AC $= (0, 2, 3)$

$$AB \times AC = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix}$$

$$= i(3 - 2) - j\,(3 - 0) + k(2 - 0)$$

$$= i - 3j + 2k$$

Area of the triangle ABC $= \frac{1}{2}\,|AB \times AC| \qquad = \frac{1}{2}\sqrt{1^2 + (-3)^2 + 2^2}$

$$= \frac{\sqrt{14}}{2} \text{ square unit}$$

- **Find the vector whose magnitude is 5 and is in the direction of $4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$:**

Given vector: $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

Magnitude:

$$\|\mathbf{v}\| = \sqrt{4^2 + (-3)^2 + 1^2} = \sqrt{16 + 9 + 1} = \sqrt{26}.$$

Unit vector:

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{26}}(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

Vector of magnitude 5:

$$\mathbf{u} = 5 \cdot \hat{\mathbf{v}} = \frac{5}{\sqrt{26}}(4\mathbf{i} - 3\mathbf{j} + \mathbf{k}).$$

`. Find the lengths of the sides of the triangle whose vertices are $A(2, 4, -1), B(4, 5, 1), C(3, 6, -3)$ and show that it is right-angled.

Use the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Side AB:

$$AB = \sqrt{(4-2)^2 + (5-4)^2 + (1-(-1))^2} = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{4+1+4} = \sqrt{9} = 3.$$

Side BC:

$$BC = \sqrt{(3-4)^2 + (6-5)^2 + (-3-1)^2} = \sqrt{(-1)^2 + 1^2 + (-4)^2} = \sqrt{1+1+16} = \sqrt{18}.$$

Side AC:

$$AC = \sqrt{(3-2)^2 + (6-4)^2 + (-3-(-1))^2} = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3.$$

Check if it's right-angled:

The square of the longest side $BC^2 = 18$ equals the sum of the squares of the other two sides:

$$AB^2 + AC^2 = 3^2 + 3^2 = 9 + 9 = 18.$$

Thus, the triangle is right-angled at A.

- HW

For what value of m, the vectors $4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $m\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ have the same magnitude?

Exercise

Q.1 Find $\vec{a} \cdot \vec{b}$ and $\vec{a} \times \vec{b}$

(i) $\vec{a} = 2i + 3j + 4k$ $\quad \vec{b} = i - j + k$

(ii) $\vec{a} = i + j + k$ $\quad \vec{b} = -5i + 2j - 3k$

(iii) $\vec{a} = -i - j - k$ $\quad \vec{b} = 2i + j$

2. Show that the vectors $3i - j + 7k$ and $-6i + 3j + 3k$ are at right angle to each other.

3. Find the cosine of the angle between the vectors:

(i) $\vec{a} = 2i - 8j + 3k$ $\quad \vec{b} = 4j + 3k$

(ii) $\vec{a} = i + 2j - k$ $\quad \vec{b} = -j - 2k$

(iii) $\vec{a} = 4i + 2j - k$ $\quad \vec{b} = 2i + 4j - k$

Q.4 If $\vec{a} = 3i + j - k$, $\vec{b} = 2i - j + k$ and $\vec{c} = 5i + 3k$, find $(2\vec{a} + \vec{b}) \cdot \vec{c}$.

Q.5 What is the cosine of the angle between $\overline{P_1P_2}$ and $\overline{P_3P_4}$ If $P_1(2,1,3)$, $P_2(-4, 4, 5)$, $P_3(0, 7, 0)$ and $P_4(-3, 4, -2)$?

Q.6 If $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$, prove that:

$$\vec{a} \cdot \vec{b} = \frac{1}{2}\left[\left|\vec{a} + \vec{b}\right|^2 - \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2\right]$$

Q7> Find $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})$ if $\vec{a} = i + 2j + 3k$ and $\vec{b} = 2i - j + k$.

Q8 Prove that for every pair of vectors a and b

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2$$

Q9 Find x so that a and b are perpendicular, Q

(i) $a = 2i + 4j - 7k$ and $b = 2i + 6j + xk$

(ii) $\vec{a} = xi - 2j + 5k$ and $\vec{b} = 2i - j + 3k$

Q.10 If $\vec{a} = 2i - 3j + 4k$ and $\vec{b} = 2j + 4k$

Find the component or projection of $\vec{a}$ along $\vec{b}$.

Q.11 Under what condition does the relation $(\vec{a} \cdot \vec{b})^2 = \vec{a}^2 \, \vec{b}^2$ hold for two vectors $\vec{a}$ and $\vec{b}$.

Q.12 If the vectors $3i + j - k$ and $\lambda I - 4j + 4k$ are parallel, find value of λ.

Q.13 If $\vec{a} = i - 2j + k$, $\vec{b} = i + 2j - 4k$, $\vec{c} = 2i - 3j + k$ Evaluate:

(i) $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c})$ (ii) $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$

Q.14 If $\vec{a} = i + 3j - 7k$ and $\vec{b} = 5i - 2j + 4k$. Find:

(i) $\vec{a} \cdot \vec{b}$ (ii) $\vec{a} \times \vec{b}$

(iii) Direction cosines of $\vec{a} \times \vec{b}$