Discrete Mathematics

Lecture 7

Functions II

By

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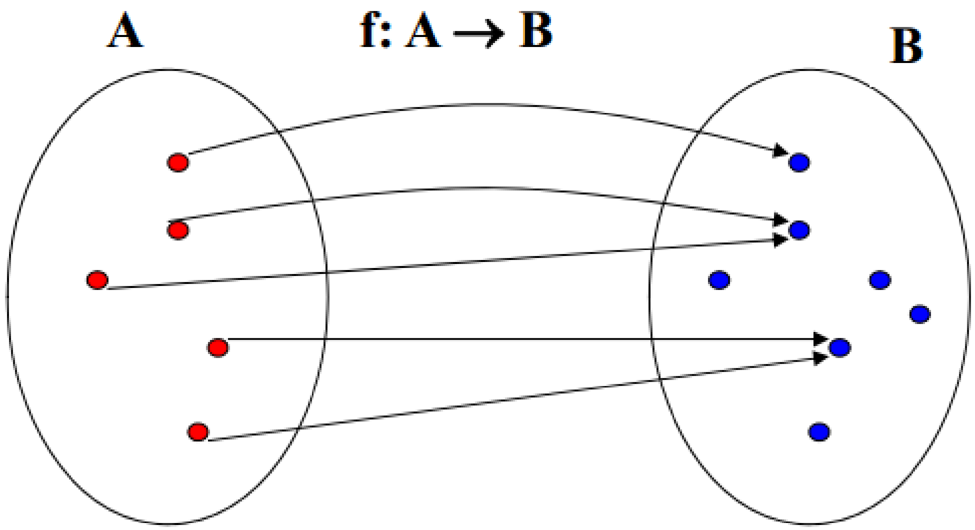
Functions

- **Definition**: Let A and B be two sets. A **function from A to B**, denoted **$f : A \rightarrow B$**, is an assignment of exactly one element of B to each element of A. We write f(a) = b to denote the assignment of b to an element a of A by the function f.



Functions

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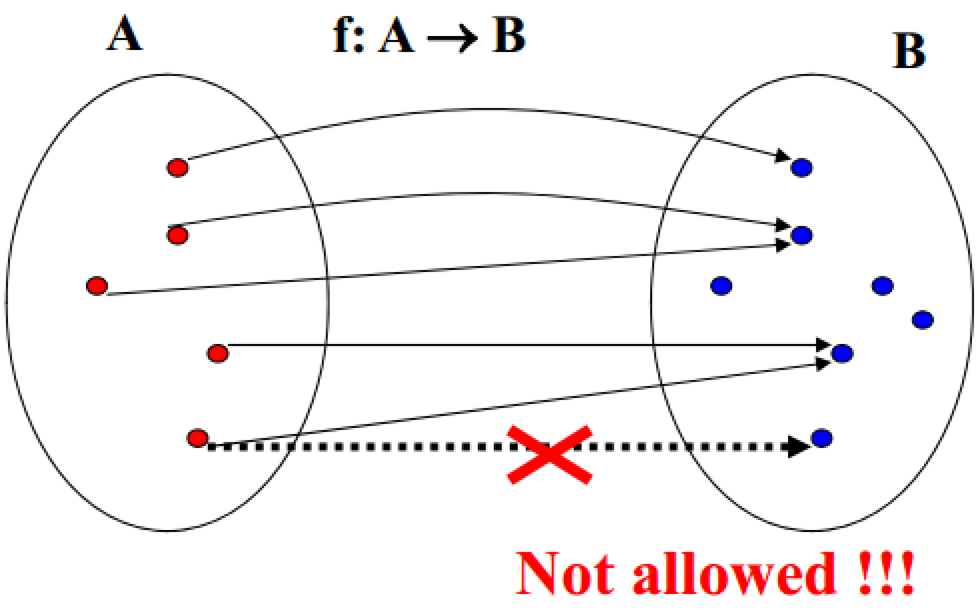
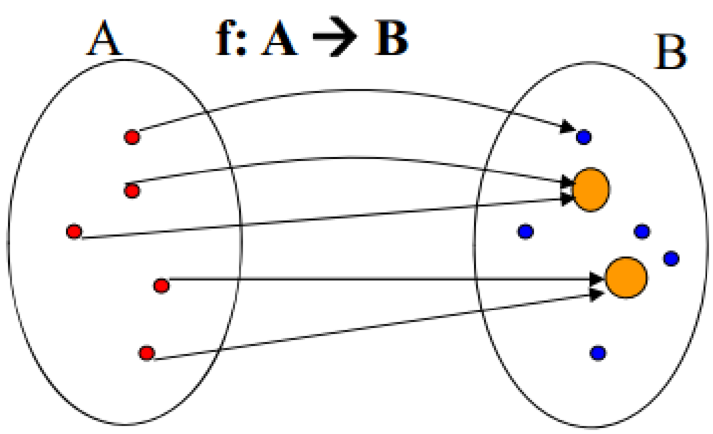


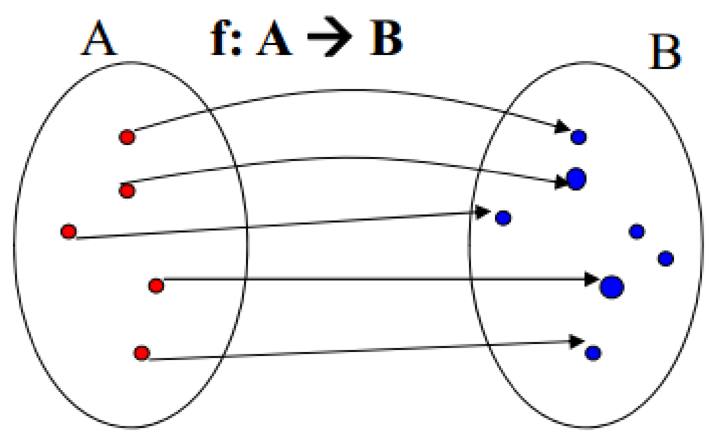
Injective function

Definition: A function f is said to be **one-to-one, or injective**, if and only if f(x) = f(y) implies x = y for all x, y in the domain of f. A function is said to be an **injection if it is one-to-one**.

Alternative: A function is one-to-one if and only if f(x) ≠ f(y), whenever x ≠ y. This is the contrapositive of the definition.



Not injective function

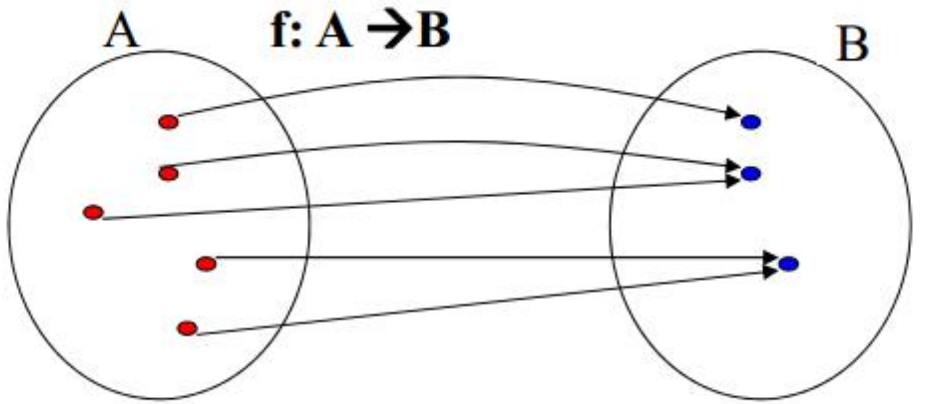


Injective function

Surjective function

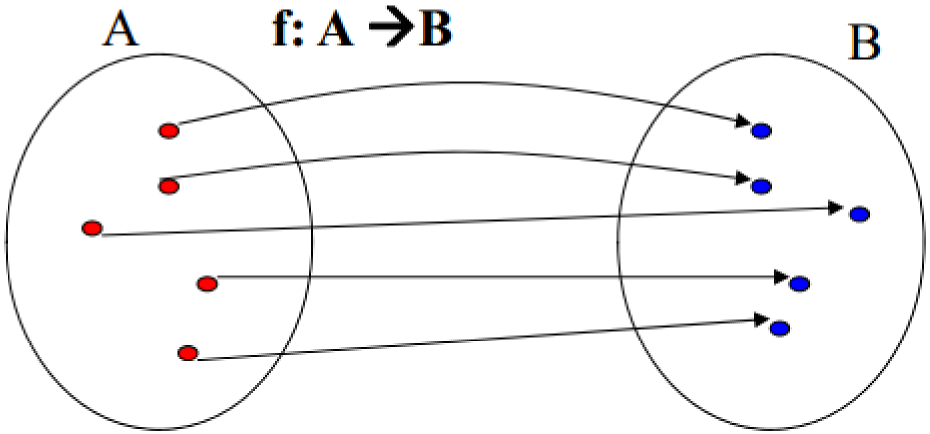
Definition: A function f from A to B is called **onto**, or **surjective**, if and only if for every b ∈ B there is an element a ∈ A such that f(a) = b.

Alternative: all co-domain elements are covered



Bijective functions

Definition: A function f is called **a bijection** if it is **both one-to-one (injection) and onto (surjection)**.



Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f a bijection?
- **?**

Bijective functions

Example 1:

- Let A = {1,2,3} and B = {a,b,c}
  - Define f as
    - $1 \rightarrow c$
    - $2 \rightarrow a$
    - $3 \rightarrow b$
- Is f a bijection?
- **Yes.** It is both one-to-one and onto.

Bijective functions

Example 2:

- Define g : W → W (whole numbers), where $g(n) = \lfloor n/2 \rfloor$ (floor function).
  - $0 \rightarrow \lfloor 0/2 \rfloor = \lfloor 0 \rfloor = 0$
  - $1 \rightarrow \lfloor 1/2 \rfloor = \lfloor 1/2 \rfloor = 0$
  - $2 \rightarrow \lfloor 2/2 \rfloor = \lfloor 1 \rfloor = 1$
  - $3 \rightarrow \lfloor 3/2 \rfloor = \lfloor 3/2 \rfloor = 1$
- ...
- Is g a bijection?

Bijective functions

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  - $3 \rightarrow \lfloor 3/2 \rfloor = \lfloor 3/2 \rfloor = 1$
- ...
- Is g a bijection?
  - **No.** g is onto but not 1-1 ($g(0) = g(1) = 0$ however $0 \neq 1$.

Bijective functions

Theorem: Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Assume

→ **A is finite** and **f is one-to-one (injective)**

- Is f an **onto function (surjection)**?

Bijective functions

Theorem: Let f be a function f: A → A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

→ **A is finite** and **f is one-to-one (injective)**

- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)

← **A is finite and f is an onto function**

- Is the function one-to-one?

Bijective functions

Theorem: Let f be a function f: A →A from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Proof:

→ **A is finite** and **f is one-to-one (injective)**

- Is f an onto function (surjection)?
- **Yes.** Every element points to exactly one element. Injection assures they are different. So we have |A| different elements A points to. Since f: A → A the co-domain is covered thus the function is also a surjection (and a bijection)

← **A is finite and f is an onto function**

- Is the function one-to-one?
- **Yes.** Every element maps to exactly one element and all elements in A are covered. Thus the mapping must be one-to-one

Bijective functions

Theorem. Let f be a function from a set A to itself, where A is finite. Then f is one-to-one if and only if f is onto.

Please note the above is not true when A is an infinite set.

- **Example:**
  - $f : Z \rightarrow Z$, where $f(z) = 2 * z$.
  - f is one-to-one but not onto.
    - $1 \rightarrow$ **2**
    - $2 \rightarrow$ **4**
    - $3 \rightarrow$ **6**
  - 3 has no pre-image.

Functions on real numbers

Definition: Let f1 and f2 be functions from A to **R** (reals). Then f1 + f2 and f1 * f2 are also functions from A to **R** defined by

- $(f1 + f2)(x) = f1(x) + f2(x)$
- $(f1 * f2)(x) = f1(x) * f2(x)$.

Examples:

- **Assume**
  - $f1(x) = x - 1$
  - $f2(x) = x^3 + 1$

  then
  - $(f1 + f2)(x) = x^3 + x$
  - $(f1 * f2)(x) = x^4 - x^3 + x - 1$.

Any questions ??