Tangent planes and normal lines

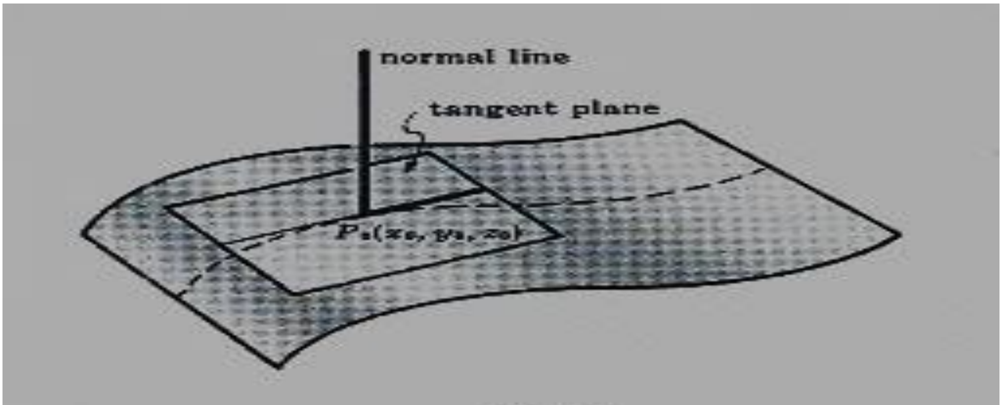

Fig. 1

The **tangent plane** at the point $P_o(x_o, y_o, z_o)$ on the level surface $f(x, y, z) = C$ of a differentiable function f is the plane through P_o normal to $\nabla f|_{P_o}$

The **normal line** of the surface at P_o is the line through P_o parallel to $\nabla f|_{P_o}$

The tangent plane and normal line have the following equation:

Tangent plane to $f(x, y, z) = C$ at $P_o(x_o, y_o, z_o)$:

$$f_x(p_o)(x - x_o) + f_y(p_o)(y - y_o) + f_z(p_o)(z - z_o) = 0$$

Normal line to $f(x, y, z) = C$ at $P_o(x_o, y_o, z_o)$:

$$x = x_o + f_x(p_o)t \;,\quad y = y_o + f_y(p_o)t \;,\quad z = z_o + f_z(p_o)t$$

Example: find the tangent plane and normal line of the surface $f(x,y,z)=x^2+y^2+z-9=0$ at the point $P_o(1,2,4)$

Solution: the tangent plane is:

$$f_x(p_o)(x-x_o)+f_y(p_o)(y-y_o)+f_z(p_o)(z-z_o)=0$$

$$f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z-9)=2x$$

$$f_x(P_o)=f_x(1,2,4)=(2)(1)=\boxed{2}$$

$$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2+z-9)=2y$$

$$f_y(P_o)=f_y(1,2,4)=(2)(2)=\boxed{4}$$

$$f_z=\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}(x^2+y^2+z-9)=1$$

$$f_z(P_o)=f_z(1,2,4)=\boxed{1}$$

$\therefore$ The tangent plane is:

$$2(x-1)+4(y-2)+(z-4)=0$$

or

$2x-2+4y-8+z-4=0$

$2x+4y+z-14=0$

$2x+4y+z=14$

The normal line is:

$$x=x_o+f_x(p_o)t \;,\; y=y_o+f_y(p_o)t \;,\; z=z_o+f_z(p_o)t$$

$$\therefore\; x=1+2t \quad , \quad y=2+4t \quad , \quad z=4+t$$

Example: find the tangent plane and normal line of the surface $f(x,y,z)=x^2+y^2+z^2=3$ at the point $P_o(1,1,1)$

Solution: the tangent plane is:

$$f_x(p_o)(x-x_o)+f_y(p_o)(y-y_o)+f_z(p_o)(z-z_o)=0$$

$f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2+z^2)=2x$

$f_x(P_o)=f_x(1,1,1)=(2)(1)=\boxed{2}$

$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(x^2+y^2+z^2)=2y$

$f_y(P_o)=f_y(1,1,1)=(2)(1)=\boxed{2}$

$f_z=\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}(x^2+y^2+z^2)=2z$

$f_z(P_o)=f_z(1,1,1)=(2)(1)=\boxed{2}$

$\therefore$ The tangent plane is:

$$\boxed{2(x-1)+2(y-1)+2(z-1)=0}$$

or

$$2x-2+2y-2+2z-2=0$$
$$2x+2y+2z-6=0$$
$$2x+2y+2z=6$$
$$2(x+y+z)=6 \Longrightarrow x+y+z=3$$

The normal line is:

$$x=x_o+f_x(p_o)t \quad , \quad y=y_o+f_y(p_o)t \quad , \quad z=z_o+f_z(p_o)t$$
$$\therefore \quad x=1+2t \quad , \quad y=1+2t \quad , \quad z=1+2t$$

Exercise:

H.W: find the tangent plane and normal line of the surface

$f(x,y,z)=x^2+2xy-y^2+z^2=7$ at the point $P_0(1,-1,3)$