



جامعة المستنقب
AL MUSTAQBAL UNIVERSITY

Discrete Mathematics

Lecture 8

Sequences and summations

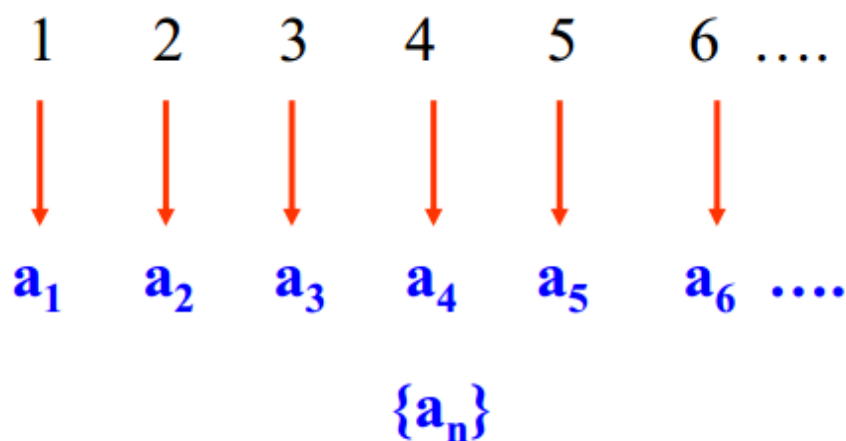
By

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Sequences

Definition: A **sequence** is a **function** from a subset of the set of integers (typically the set $\{0,1,2,\dots\}$ or the set $\{1,2,3,\dots\}$ to a set S . We use the notation a_n to denote the image of the integer n . We call a_n a term of the sequence.

Notation: $\{a_n\}$ is used to represent the sequence (note $\{\}$ is the same notation used for sets, so be careful). $\{a_n\}$ represents the ordered list a_1, a_2, a_3, \dots .



Sequences

Examples:

- (1) $a_n = n^2$, where $n = 1, 2, 3, \dots$
 - What are the elements of the sequence?
 $1, 4, 9, 16, 25, \dots$
- (2) $a_n = (-1)^n$, where $n = 0, 1, 2, 3, \dots$
 - Elements of the sequence?
 $1, -1, 1, -1, 1, \dots$
- 3) $a_n = 2^n$, where $n = 0, 1, 2, 3, \dots$
 - Elements of the sequence?
 $1, 2, 4, 8, 16, 32, \dots$

Arithmetic progression

Definition: An **arithmetic progression** is a sequence of the form

$$a, a+d, a+2d, \dots, a+nd$$

where a is the *initial term* and d is *common difference*, such that both belong to \mathbb{R} .

Example:

- $s_n = -1 + 4n$ for $n = 0, 1, 2, 3, \dots$
- members: $-1, 3, 7, 11, \dots$

Geometric progression

Definition A **geometric progression** is a sequence of the form:

$$a, ar, ar^2, \dots, ar^k,$$

where a is the *initial term*, and r is the *common ratio*. Both a and r belong to \mathbb{R} .

Example:

- $a_n = \left(\frac{1}{2}\right)^n$ for $n = 0, 1, 2, 3, \dots$
members: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$

Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward

Example:

- Assume the sequence: 1,3,5,7,9,
- What is the formula for the sequence?
- Each term is obtained by adding 2 to the previous term.
1, $1+2=3$, $3+2=5$, $5+2=7$
- What type of progression this suggest?

Sequences

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- What is the formula for the sequence?
- Each term is obtained by adding 2 to the previous term.
- 1, $1+2=3$, $3+2=5$, $5+2=7$
- It suggests **an arithmetic progression**: $a+nd$
with $a=1$ and $d=2$
 - $a_n=1+2n$

Sequences

- Given a sequence finding a rule for generating the sequence is not always straightforward

Example 2:

- Assume the sequence: $1, 1/3, 1/9, 1/27, \dots$
- What is the sequence?
- The denominators are powers of 3.
 $1, 1/3 = 1/3, (1/3)/3 = 1/(3*3) = 1/9, (1/9)/3 = 1/27$
- This suggests a **geometric progression**: ar^k
with $a=1$ and $r=1/3$
 - $(1/3)^n$

Recursively defined sequences

- The n -th element of the sequence $\{a_n\}$ is defined recursively in terms of the previous elements of the sequence and the initial elements of the sequence.

Example :

- $a_n = a_{n-1} + 2$ assuming $a_0 = 1$;
- $a_0 = 1$;
- $a_1 = 3$;
- $a_2 = 5$;
- $a_3 = 7$;
- Can you write a_n non-recursively using n ?
- $a_n = 1 + 2n$

Fibonacci sequence

- Recursively defined sequence, where
 - $f_0 = 0$;
 - $f_1 = 1$;
 - $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, \dots$
-
- $f_2 = 1$
 - $f_3 = 2$
 - $f_4 = 3$
 - $f_5 = 5$

Summations

Summation of the terms of a sequence:

$$\sum_{j=m}^n a_j = a_m + a_{m+1} + \dots + a_n$$

The variable j is referred to as the index of summation.

- m is the *lower limit* and
- n is the *upper limit* of the summation.

Summations

Example:

- 1) Sum the first 7 terms of $\{n^2\}$ where $n=1,2,3, \dots$.

- $$\sum_{j=1}^7 a_j = \sum_{j=1}^7 j^2 = 1 + 4 + 16 + 25 + 36 + 49 = 140$$

- 2) What is the value of

$$\sum_{k=4}^8 a_j = \sum_{k=4}^8 (-1)^j = 1 + (-1) + 1 + (-1) + 1 = 1$$

Arithmetic series

Definition: The sum of the terms of the arithmetic progression $a, a+d, a+2d, \dots, a+nd$ is called an **arithmetic series**.

Theorem: The sum of the terms of the arithmetic progression $a, a+d, a+2d, \dots, a+nd$ is

$$S = \sum_{j=1}^n (a + jd) = na + d \sum_{j=1}^n j = na + d \frac{n(n+1)}{2}$$

- **Why?**

Arithmetic series

Theorem: The sum of the terms of the arithmetic progression
 $a, a+d, a+2d, \dots, a+nd$ is

$$S = \sum_{j=1}^n (a + jd) = na + d \sum_{j=1}^n j = na + d \frac{n(n+1)}{2}$$

Proof:

$$S = \sum_{j=1}^n (a + jd) = \sum_{j=1}^n a + \sum_{j=1}^n jd = na + d \sum_{j=1}^n j$$

$$\sum_{j=1}^n j = 1 + 2 + 3 + 4 + \dots + (n-2) + (n-1) + n$$

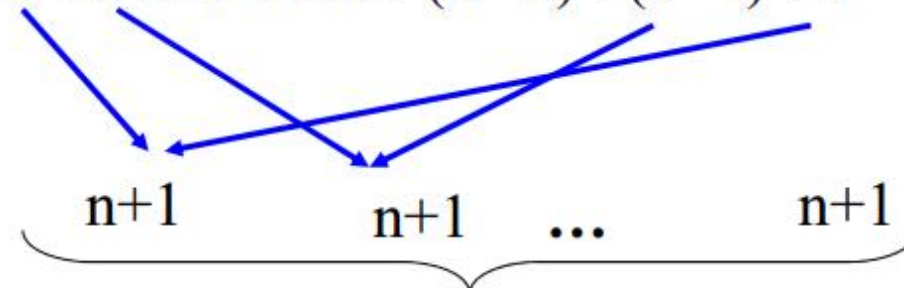
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$$\sum_{j=1}^n j = 1 + 2 + 3 + 4 + \dots + (n-2) + (n-1) + n$$


$n+1 \quad n+1 \quad \dots \quad n+1$

$\frac{n}{2} * (n+1)$

Arithmetic series

Example:

$$\begin{aligned} S &= \sum_{j=1}^5 (2 + j3) = \\ &= \sum_{j=1}^5 2 + \sum_{j=1}^5 j3 = \\ &= 2 \sum_{j=1}^5 1 + 3 \sum_{j=1}^5 j = \\ &= 2 * 5 + 3 \sum_{j=1}^5 j = \\ &= 10 + 3 \frac{(5+1)}{2} * 5 = \\ &= 10 + 45 = 55 \end{aligned}$$

Arithmetic series

Example 2:

$$\begin{aligned} S &= \sum_{j=3}^5 (2 + j3) = \\ &= \left[\sum_{j=1}^5 (2 + j3) \right] - \left[\sum_{j=1}^2 (2 + j3) \right] \quad \leftarrow \text{Trick} \\ &= \left[2 \sum_{j=1}^5 1 + 3 \sum_{j=1}^5 j \right] - \left[2 \sum_{j=1}^2 1 + 3 \sum_{j=1}^2 j \right] \\ &= 55 - 13 = 42 \end{aligned}$$

Double summations

Example: $S = \sum_{i=1}^4 \sum_{j=1}^2 (2i - j) =$

$$\begin{aligned} &= \sum_{i=1}^4 \left[\sum_{j=1}^2 2i - \sum_{j=1}^2 j \right] = \\ &= \sum_{i=1}^4 \left[2i \sum_{j=1}^2 1 - \sum_{j=1}^2 j \right] = \\ &= \sum_{i=1}^4 \left[2i * 2 - \sum_{j=1}^2 j \right] = \\ &= \sum_{i=1}^4 [2i * 2 - 3] = \\ &= \sum_{i=1}^4 4i - \sum_{i=1}^4 3 = \\ &= 4 \sum_{i=1}^4 i - 3 \sum_{i=1}^4 1 = 4 * 10 - 3 * 4 = 28 \end{aligned}$$

Geometric series

Definition: The sum of the terms of a geometric progression a, ar, ar^2, \dots, ar^k is called **a geometric series**.

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, \dots, ar^n is

$$S = \sum_{j=0}^n (ar^j) = a \sum_{j=0}^n r^j = a \left[\frac{r^{n+1} - 1}{r - 1} \right]$$

Geometric series

Theorem: The sum of the terms of a geometric progression a, ar, ar^2, \dots, ar^n is

$$S = \sum_{j=0}^n (ar^j) = a \sum_{j=0}^n r^j = a \left[\frac{r^{n+1} - 1}{r - 1} \right]$$

Proof:

$$S = \sum_{j=0}^n ar^j = a + ar + ar^2 + ar^3 + \dots + ar^n$$

- multiply S by r

$$rS = r \sum_{j=0}^n ar^j = ar + ar^2 + ar^3 + \dots + ar^{n+1}$$

- Subtract $rS - S = [ar + ar^2 + ar^3 + \dots + ar^{n+1}] - [a + ar + ar^2 + \dots + ar^n]$
 $= ar^{n+1} - a$



$$S = \frac{ar^{n+1} - a}{r - 1} = a \left[\frac{r^{n+1} - 1}{r - 1} \right]$$

Geometric series

Example:

$$S = \sum_{j=0}^3 2(5)^j =$$

General formula:

$$S = \sum_{j=0}^n (ar^j) = a \sum_{j=0}^n r^j = a \left[\frac{r^{n+1} - 1}{r - 1} \right]$$

$$S = \sum_{j=0}^3 2(5)^j = 2 * \frac{5^4 - 1}{5 - 1} =$$

$$= 2 * \frac{625 - 1}{4} = 2 * \frac{624}{4} = 2 * 156 = 312$$

Infinite geometric series

- Infinite geometric series can be computed in the closed form for $x < 1$
- **How?**

$$\sum_{n=0}^{\infty} x^n = \lim_{k \rightarrow \infty} \sum_{n=0}^k x^n = \lim_{k \rightarrow \infty} \frac{x^{k+1} - 1}{x - 1} = -\frac{1}{x - 1} = \frac{1}{1 - x}$$

- **Thus:**

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

*Any
questions ??*