عنوان المحاضرة:

Double Intergrals, Areas التكاملات الثنائية، المساحات

Multiple Integrals

The multiple integrals are the integrals of a function of two or more variables over a region in the plane or space.

Double Integrals :-

(1) If a function $f(x,y)$ is defined on a rectangular region R

R: $a \leq x \leq b$ & $c \leq y \leq d$, then

$$\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

(2) If a function $f(x,y)$ is defined on a region R

R: $a \leq x \leq b$ & $f_1(x) \leq y \leq f_2(x)$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\, dy\, dx$$

(3) If a function $f(x,y)$ is defined on a region R

R: $g_1(x) \leq x \leq g_2(x)$ & $c \leq y \leq d$

$$\iint_R f(x,y)\, dA = \int_c^d \int_{g_1(x)}^{g_2(x)} f(x,y)\, dx \cdot dy$$

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Ex:- Calculate $\iint_R f(x,y)\,dA$ for

$f(x,y) = 1-6x^2y$ & $R: 0\leq x\leq 2, -1\leq y\leq 1$

Solution:-

$$\iint_R f(x,y)\,dA = \int_0^2\int_{-1}^1 (1-6x^2y)\,dy\,dx = \int_0^2 \left[y - \frac{6x^2y^2}{2}\right]_{-1}^1 dx$$

$$= \int_0^2 \left[\left(1-\frac{6x^2*1^2}{2}\right)-\left(-1-\frac{6x^2(-1)^2}{2}\right)\right]dx$$

$$= \int_0^2 [1-3x^2-(-1-3x^2)]\,dx = \int_0^2 2\,dx = 2x\Big]_0^2$$

$$= (2*2)-(2*0) = \boxed{4}$$

Ex:- Calculate $\int_0^1\int_0^y e^{y^2}\,dx\cdot dy$.

Solution:-

$$\int_0^1\int_0^y e^{y^2}\,dx\cdot dy = \int_0^1 e^{y^2}\cdot x\Big]_0^y dy = \int_0^1 [(e^{y^2}*y)-(e^{y^2}*0)]\cdot dy$$

$$= \int_0^1 y e^{y^2} dy * \frac{2}{2} = \frac{1}{2} e^{y^2}\Big]_0^1 = \frac{1}{2}(e^{1^2}-e^0)$$

$$= \boxed{\frac{1}{2}(e^1-1)}$$

Ex:- Find $\int_0^2\int_{y^2}^{6-y} dx\cdot dy$

Solution:-

$$\int_0^2\int_{y^2}^{6-y} dx\cdot dy = \int_0^2 x\Big]_{y^2}^{6-y} dy = \int_0^2 (6-y-y^2)\,dy = 6y-\frac{y^2}{2}-\frac{y^3}{3}\Big]_0^2$$

$$= \left(6*2-\frac{2^2}{2}-\frac{2^3}{3}\right)-(0) = 12-2-\frac{8}{3}$$

$$= \frac{22}{3} = \boxed{7.33}$$

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EX:- Calculate $\iint_R f(x,y)\,dA$ for

$f(x,y) = 4x+2$ & $R: x^2 \leq y \leq 2x$, $0 \leq x \leq 2$

Solution:-

$$\iint_R f(x,y)\,dA = \int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx = \int_0^2 (4xy+2y)\Big]_{y=x^2}^{y=2x} \cdot dx$$

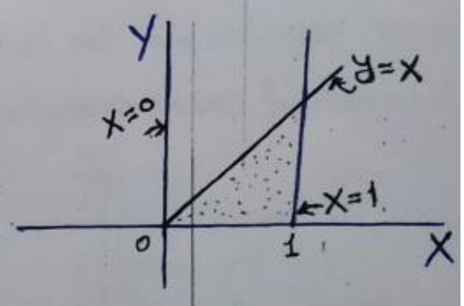
$$= \int_0^2 [(4x*2x+2*2x)-(4x*x^2+2*x^2)] \cdot dx$$

$$= \int_0^2 [8x^2+4x-(4x^3+2x^2)]dx = \int_0^2 (6x^2+4x-4x^3) \cdot dx$$

$$= \frac{6x^3}{3} + \frac{4x^2}{2} - \frac{4x^4}{4}\Big]_0^2 = 2x^3+2x^2-x^4\Big]_0^2$$

$$= 2(2)^3+2(2)^2-(2)^4 = 16+8-16 = \boxed{8}$$

EX:- Evaluate the $\iint_R \frac{\sin x}{x}\,dA$ where R is the triangle in xy-plane bounded by the x-axis the line $x=y$ and the line $x=1$.



Solution:-

$$\int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^1 \frac{\sin x}{x} y\Big]_0^{y=x} \cdot dx$$

$$\int_0^1 \frac{\sin x}{\cancel{x}} \cdot \cancel{x}\,dx = -\cos x\Big]_0^1$$

$$= -(\cos(1) - \cos(0))$$

$$= \boxed{0.459}$$

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Determining the Limits of Integration:-

If we want to evaluate $\iint_R f(x,y)\,dA$ over the region R shown in Fig.(1), integrating first with respect to (y) and then with respect to (x). we take the following steps:

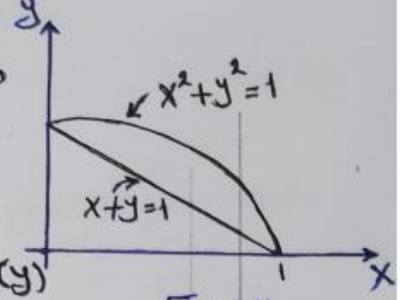

Fig.(1)

1. we imagine a vertical line L cutting through R in the direction of increasing (y) as shown in Fig.(2).

2. we integrate from the y-value where L enters R to the y-value where L leave R.

3. we choose x-limits that include all the vertical lines that pass through R.

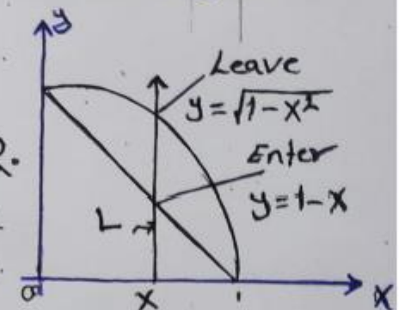

Fig.(2)

$$\therefore \iint_R f(x,y)\,dA = \int_{x=0}^{x=1}\int_{y=1-x}^{y=\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

To calculate the same double integral as an integral with the order of integration reversed the procedure uses horizontal line as shown in Fig (3)

$$\therefore \iint_R f(x,y)\,dA = \int_{y=0}^{y=1}\int_{x=1-y}^{x=\sqrt{1-y^2}} f(x,y)\,dx\,dy.$$

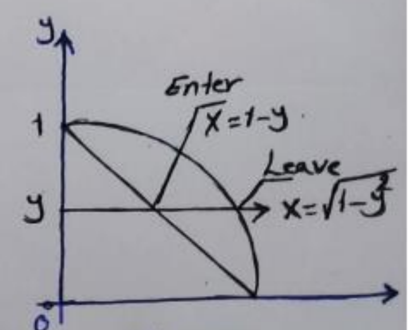

Fig (3)

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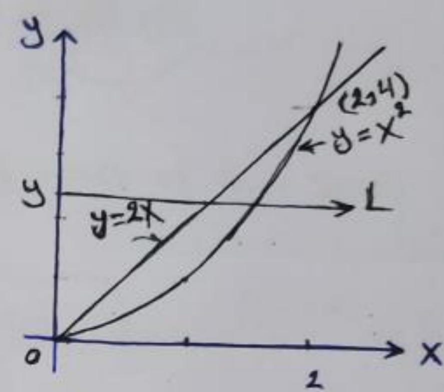
Ex:- Find an equivalent integral to the $\int_0^2\int_{x^2}^{2x}(4x+2)\,dy\,dx$ by the order of the integral reversed.

Solution:-



R: $x=0$ to $x=2$

$y=x^2$ to $y=2x$

$$\therefore \int_0^2\int_{x^2}^{2x}(4x+2)\,dy\,dx = \int_0^4\int_{\frac{y}{2}}^{\sqrt{y}}(4x+2)\,dx\,dy$$

$$\int_0^4\int_{y/2}^{\sqrt{y}}(4x+2)\,dx\,dy = \int_0^4\left(\frac{4x^2}{2}+2x\right)\Big]_{y/2}^{\sqrt{y}}\cdot dy$$

$$= \int_0^4\left[\left(2(\sqrt{y})^2+2\sqrt{y}\right)-\left(2\left(\frac{y}{2}\right)^2+2*\frac{y}{2}\right)\right]dy$$

$$= \int_0^4\left(2y+2\sqrt{y}-\frac{y^2}{2}-y\right)dy = \int_0^4\left(y+2\sqrt{y}-\frac{y^2}{2}\right)dy$$

$$= \left[\frac{y^2}{2}+\frac{2(y)^{3/2}}{3/2}-\frac{y^3}{2*3}\right]_0^4$$

$$= \left(\frac{4^2}{2}+\frac{4}{3}(4)^{3/2}-\frac{(4)^3}{6}\right)$$

$$= 8+\frac{4}{3}(8)-\frac{32}{3}$$

$$= \boxed{8}$$

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