عنوان المحاضرة:

## Maxim and Minima for two variables functions القيم العظمى والصغرى لدوال في متغيرين

### Maxima, Minima and Saddle Points :-

If $f(x,y)$ is a function of two independent variables $(x,y)$ and the interior points $(a,b)$ are found at $f_x = f_y = 0$. then:

① $f$ has a local maximum at $(a,b)$ if $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at $(a,b)$

② $f$ has a local minimum at $(a,b)$ if $f_{xx} > 0$ and $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$ at $(a,b)$

③ $f$ has a saddle point at $(a,b)$ if $f_{xx} f_{yy} - f_{xy}^2 < 0$ at $(a,b)$ where $(a,b)$ is the critical point

**Ex:-** Find the extreme value of $f(x,y) = x^2 + y^2$

**Solution:-**

$f_x = 2x \quad , \quad f_x = 0 = 2x \Rightarrow x = 0 = a$

$f_y = 2y \quad , \quad f_y = 0 = 2y \Rightarrow y = 0 = b$

$f_{xx} = 2 \quad , \quad f_{yy} = 2 \quad , \quad f_{xy} = 0 \Rightarrow f_{xy}^2 = 0$

$f_{xx} f_{yy} - f_{xy}^2 = 2 * 2 - 0 = 4 > 0$

$\because f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

and $(a,b) = (0,0)$ is the critical point

$\therefore f(x,y) = x^2 + y^2$ has a local minimum at $(a,b)$

$f(0,0) = 0$

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**Ex:-** Find the extreme value of the function

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

**Solution:-**

$f_x = y - 2x - 2 \Rightarrow \boxed{f_{xx} = -2}$

$f_x = 0 \Rightarrow y - 2x - 2 = 0$ —— ①

$f_y = x - 2y - 2 \Rightarrow \boxed{f_{yy} = -2}$

$f_y = 0 \Rightarrow x - 2y - 2 = 0$ —— ②

لايجاد قيمة x و y من المعادلتين (1 و 2) يتم حل المعادلتين آنياً

$y - 2x - 2 = 0$ —— ①
$-2y + x - 2 = 0$ —— ② $*2$

$y - 2x - 2 = 0$ —— ①
$-4y + 2x - 4 = 0$ —— ②

بالجمع

$-3y - 6 = 0 \Rightarrow 3y = -6 \Rightarrow y = -2 = b$

نعوض قيمة y في المعادلة (1) لايجاد قيمة x

$-2 - 2x - 2 = 0 \Rightarrow -4 = 2x$

$x = -2 = a$

$\therefore (a,b) = (-2,-2)$

$f_{xx} f_{yy} - f_{xy}^2$

$-2 * -2 - (1)^2 = 3 > 0$

$* f_{xx} = -2$
$* f_{yy} = -2$

$\because f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

$\therefore f(x,y)$ has a local maximum at $(-2,-2)$

$f(-2,-2) = 8$

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## Lagrange Multipliers :-

The extreme values of a function $f(x,y,z)$ whose variable are subject to a constraint of form $g(x,y,z)=0$ are to be found on the surface $g=0$ at the points where

$$\nabla f = \lambda \nabla g$$

$\lambda$ called a Lagrange multiplier.

**Ex:-** Find the greatest and smallest values that the function $f(x,y) = xy$ takes on the ellipes

$$\frac{x^2}{8} + \frac{y^2}{2} = 1$$

**Solution:-**

$$g(x) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$$

$$\nabla f = \lambda \nabla g$$

$$\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j = \lambda \left( \frac{\partial g}{\partial x} i + \frac{\partial g}{\partial y} j \right)$$

$$y i + x j = \lambda \left( \frac{2x}{8} i + \frac{2y}{2} j \right)$$

$$y = \lambda \frac{x}{4} \quad \text{—} \; (1)$$

$$x = \lambda y \quad \text{—} \; (2) \quad \text{sub in eq (1)}$$

$$y = \lambda * \frac{\lambda y}{4} = \frac{\lambda^2 y}{4}$$

$$\frac{\lambda^2}{4} = 1 \Rightarrow \lambda^2 = 4 \Rightarrow \boxed{\lambda = \mp 2}$$

or $y = 0$



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**Case I :-** If $y=0$ then $x=y=0$

The point $(0,0)$ is not on the ellipes

**Case II :-** If $y\neq 0$ then $\lambda=\mp 2$

$$x=\mp 2y$$

Sub. this $(x=\mp 2y)$ in the eq $g(x,y)=0$ gives

$$\frac{(\mp 2y)^2}{8}+\frac{y^2}{2}=1$$

$$\frac{4y^2+4y^2}{8}=1 \Rightarrow 8y^2=8$$

$$y^2=1 \Rightarrow y=\mp 1$$

$$f(x,y)=f(\mp 2,\mp 1)$$

$$f(\mp 2,1)=\mp 2$$

$$f(\mp 2,-1)=\mp 2$$

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H.W

(1) Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if $f(x,y)=x^2+3xy+y-1$

(2) Find $\partial f/\partial x$ and $\partial f/\partial y$ for $f(x,y)=(x^2-1)(y+2)$

(3) Find $\partial f/\partial x$ and $\partial f/\partial y$ for $f(x,y)=\frac{x}{x^2+y^2}$

(4) Find $f_x$, $f_y$ and $f_z$ for

a/ $f(x,y,z)=\ln(x+2y+3z)$

b/ $f(x,y,z)=e^{-(x^2+y^2+z^2)}$

(5) Find all the second order partial derivatives of the $f(x,y)=\sin(xy)$

(6) Find the value of $df/dt$ at $t=0$ if $f(x,y)=x^2+y^2$ and $x=\cos t+\sin t$, $y=\cos t-\sin t$.

(7) Find the value of $df/dt$ at $t=3$ if $f(x,y,z)=\ln(x^2+y^2+z^2)$ and $x=\cos t$, $y=\sin t$, $z=4\sqrt{t}$

(8) Find $\nabla f$ at the point $(1,1,1)$ if $f(x,y,z)=x^2+y^2-2z^2+z\ln x$

(9) Find the derivative of $f(x,y,z)=xy+yz+zx$ at $P_0(1,-1,2)$ in the direction of $\vec{v}=3i+6j-2k$.

(10) Find the local minima, local maxima and saddle points of $f(x,y)=x^2+2xy$

(11) Find the maximum values of $f(x,y)=49-x^2-y^2$ on the line $x+3y=10$



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