Al-Mustaqbal university

Engineering technical college

Department of Building &Construction Engineering

Mathematics

First class

Lecture no.2

Assist. Lecture

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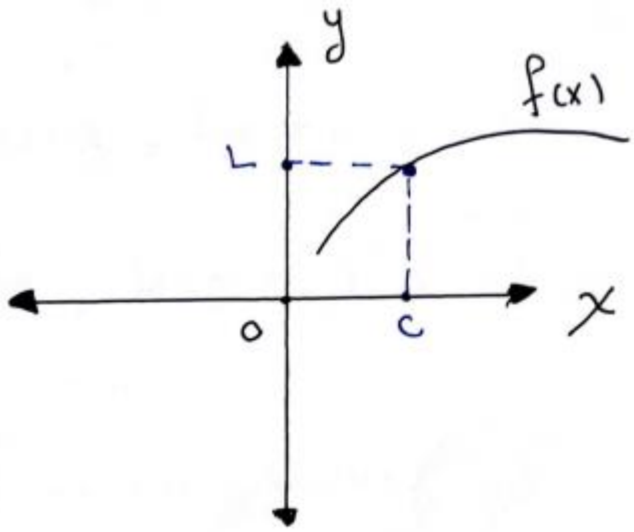
Limits:

When the values of a function f(x) approach the value L as X Approaches C, we say that F(x)has limit L as X Approaches C.

OR: $\lim_{X \to C} f(x) = L$



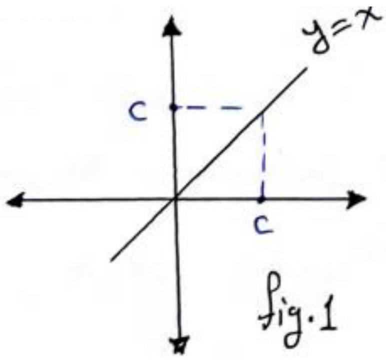
*Notes:

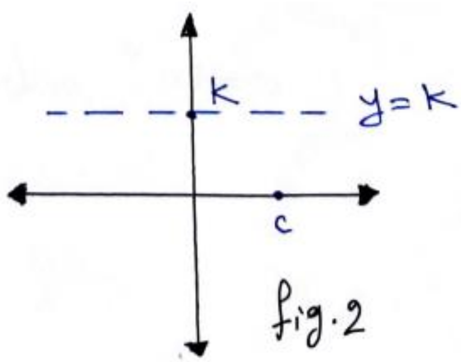
1. For identify function (f(x)=x)

$$\lim_{X \to C} x = c$$

2. For constant function (f(x)=k)

3. $\lim_{X \to C} k = k$





Properties of limits:

For

$$\lim_{X\to C} F_1(x) = L_1 \quad \& \quad \lim_{X\to C} F_2(x) = L_2$$

1. $\lim_{X\to C}[\ F_1(x) \pm F_2(x)\ ] = L_1 \ \pm \ \ L_2$
2. $\lim_{X\to C}[\ F_1(x) \times F_2(x)\ ] = L_1 \ \times \ \ L_2$
3. $\lim_{X\to C}[\ F_1(x) \div F_2(x)\ ] = L_1 \ \div \ \ L_2 \qquad\qquad L_2 \neq 0$
4. $\lim_{X\to C}[\ F_1(x) \times k\ ] = k \ \times \ \ L_1 \qquad\qquad k: constant$

- If $F_1(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots \ldots \ldots a_0 \quad is\ any\ polynomial\ function$:
- $\lim_{X\to C} f(x) = f(c) = a_n C^n + a_{n-1}C^{n-1} + \cdots \ldots \ldots a_0$

Ex (1): find the limit of the function f(x)=x+1 as x approaches 3?

Sol/

f(x)=x+1

$$\lim_{X\to C} f(x) = \lim_{X\to 3} x + 1$$

$$\lim_{X\to 3} x + \lim_{X\to 3} 1$$

$$3 + 1 = 4$$

Ex (2):

$$\lim_{X\to 3} \frac{x^3 - 27}{x - 3}$$

$$\frac{(0)^3 - 27}{0 - 3} = \frac{0}{0}$$

$$\lim_{X\to 3} \frac{(x-3)(x^2+3x+9)}{(x-3)}$$

$$\lim_{X\to 3}(x^2 + 3x + 9)$$

$$(3^2 + 3(3) + 9) = 27$$

Ex (3): Evaluate

$$\lim_{X\to 1}\frac{x^2+x-2}{x^2-x}$$

$$\frac{1^2+1-2}{1^2-1}=\frac{0}{0}$$

$$\lim_{X\to 1}\frac{(x+2)(x-1)}{x(x-1)}$$

$$\lim_{X\to 1}\frac{(x+2)}{x}$$

$$\frac{1+2}{1}=3$$

Ex (4) find the limits:

$$\lim_{X\to 5}\frac{4}{x-7}$$

$$\frac{4}{5-7}=\frac{4}{-2}=-2$$

Ex (5) find the limits:

$$\lim_{X\to 2}\frac{x^2-7x-10}{x-2}$$

$$\lim_{X\to 2}\frac{(x-2)(x-5)}{(x-2)}$$

$$\lim_{X\to 2}(x-5)=2-5=-3$$

EX (6):

$$f(x) = \begin{cases} 3 - x & .x < 2 \\ \frac{x}{2} + 1 & .x > 2 \end{cases}$$

$$a - find \lim_{X\to 2} f(x) and \lim_{X\to -2} f(x)$$

$$b - dose \lim_{X\to -2} f(x)\ exist?\ why$$

Sol/

$$a - \lim_{X\to 2} f(x) = \lim_{X\to 2} \frac{x}{2} + 1 \quad = \frac{2}{2} + 1 = 2$$

$$b - \lim_{X\to -2} f(x) = \lim_{X\to -2} (3 - x) = 3 - 2 = 1$$

$$\lim_{X\to 2} f(x) \neq \lim_{X\to -2} f(x)$$

Limit does not exist

Ex (7):

$$\lim_{X\to 1} \frac{x^3 - 1}{(x-1)^2}$$

$$\lim_{X\to 1} \frac{(x-1)(x^2 + x + 1)}{(x-1)\,(x-1)}$$

$$\lim_{X\to 1} \frac{(x^2 + x + 1)}{(x-1)}$$

H.W/ Let:

$$f(x) = \begin{cases} 3 - x & .x < 2 \\ \frac{x}{2} & .x > 2 \end{cases}$$

a- Find $\lim_{X\to 2} f(x) and \lim_{X\to -2} f(x)$

$b - dose \lim_{X\to -2} f(x)\ exist?\ why$

Ex : find $\lim_{X\to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x^3+8}$

$$\lim_{X\to -2} \frac{\frac{(x+2)}{2x}}{(x+2)(x^2-2x+4)}$$

$$\lim_{X\to -2} \frac{1}{2x(x^2-2x+4)}$$

$$= \frac{1}{2(-2)((-2)^2-2(-2)+4)}$$

$$= \frac{-1}{48}$$

Ex:

$$\lim_{X\to 4} \frac{3-\sqrt{x+5}}{x-4}$$

$$\lim_{X\to 4} \frac{3-\sqrt{x+5}}{x-4} * \frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}$$

$$\lim_{X\to 4} \frac{9-(x+5)}{x-4(3+\sqrt{x+5}}$$

$$\lim_{X\to 4} \frac{4-x}{x-4(3+\sqrt{x+5}}$$

$$\lim_{X\to 4} \frac{-(x-4)}{(x-4)(3+\sqrt{x+5}}$$

$$\lim_{X\to 4} \frac{-1}{(3+\sqrt{x+5}} = \frac{-1}{(3+\sqrt{4+5}} = \frac{-1}{6}$$

The sandwiches theorem

$$f(x).h(x).g(x)\ are\ functions$$

$$if\ f(x) \leq h(x) \leq g(x)$$

The sandwiches theorem is:

For all x ≠ c and $\lim_{X\to c} g(x) = \lim_{X\to c} h(x) = L$

Then $\lim_{X\to c} f(x) = L$

Ex (1): find $\lim_{X\to 0} x^4 cos\frac{1}{x}$

$$-1 \leq cos\frac{1}{x} \leq 1$$

$$\left[-1 \leq cos\frac{1}{x} \leq 1\right] * x^4$$

$$-x^4 \leq x^4 cos\frac{1}{x} \leq x^4$$

$$\lim_{X\to 0} -x^4 \leq \lim_{X\to 0} x^4 cos\frac{1}{x} \leq \lim_{X\to 0} x^4$$

$$\lim_{X\to 0} -x^4 = (0)^4 = 0\ g(x)$$

$$\lim_{X\to 0} x^4 = (0)^4 = 0\ h(x)$$

$$0 \leq cos\frac{1}{x} \leq 0$$

h(x)= g(x)

Cross ponding to sandwiches theory

$$\lim_{X\to 0} x^4 cos\frac{1}{x} = 0\ f(x)$$

Ex (2): find $\lim_{X\to 0}(x^2 sin\frac{1}{\sqrt{x}})$

Sol/

$$-1 \le sin\frac{1}{\sqrt{x}} \le 1$$

$$\left[-1 \le sin\frac{1}{\sqrt{x}} \le 1\right] *\ x^2$$

$$-\ x^2 \le\ x^2 sin\frac{1}{\sqrt{x}} \le\ x^2$$

$$\lim_{X\to 0} -\ x^2 \le \lim_{X\to 0} x^2 sin\frac{1}{\sqrt{x}} \le \lim_{X\to 0} x^2$$

$$\lim_{X\to 0} -\ x^2 = -(0)^2 = 0\ \ g(x)$$

$$\lim_{X\to 0} x^2 = (0)^2 = 0\ \ h(x)$$

Sandwiches theory is:

$\lim_{X\to 0} x^2 sin\frac{1}{\sqrt{x}} = 0\ \ f(x)$

Ex (3): find $\lim_{X\to\infty}\frac{\cos x}{x^2+1}$

Sol/

$$-1 \le cosx \le 1$$

$$[-1 \le cosx \le 1] \div (\ x^2+1)$$

$$\frac{-1}{x^2+1} \le \frac{cosx}{x^2+1} \le \frac{1}{x^2+1}$$

$$\lim_{X\to\infty}\frac{-1}{x^2+1} \le \lim_{X\to\infty}\frac{cosx}{x^2+1} \le \lim_{X\to\infty}\frac{1}{x^2+1}$$

$$\lim_{X\to\infty}\frac{-1}{x^2+1} = \frac{-1}{\infty^2+1} = \frac{-1}{\infty^2} = 0 \ldots\ g(x)$$

$$\lim_{X\to\infty}\frac{1}{x^2+1} = \frac{1}{\infty^2+1} = \frac{1}{\infty^2} = 0 \ldots\ h(x)$$

By sandwiches theory

$$\lim_{X\to\infty} \frac{cosx}{x^2+1} = 0 \ \ldots \ f(x)$$

Ex (4):

Find $\lim_{X\to\infty} \frac{2-\cos x}{x^2+3}$ $by\ using\ sandwiches\ theorm?$

Sol/

$$-1 \le cosx \le 1$$

$$[-1 \le cosx \le 1] * -1$$

$$1 \ge -cosx \ge -1$$

$$[1 \ge -cosx \ge -1] + 2$$

$$1 + 2 \ge 2 - cosx \ge -1 + 2$$

$$[3 \ge 2 - cosx \ge +1] \div (\ x^2 + 1)$$

$$\frac{3}{x^2+1} \ge \frac{2-cosx}{x^2+1} \ge \frac{+1}{x^2+1}$$

$$\lim_{X\to\infty} \frac{3}{x^2+1} \ge \lim_{X\to\infty} \frac{2-cosx}{x^2+1} \ge \lim_{X\to\infty} \frac{+1}{x^2+1}$$

$$\lim_{X\to\infty} \frac{3}{x^2+1} = \frac{3}{\infty^2+1} = \frac{3}{\infty^2} = 0 \ldots \ldots g(x)$$

$$\lim_{X\to\infty} \frac{+1}{x^2+1} = \frac{1}{\infty^2+1} = \frac{1}{\infty^2} = 0 \ldots \ldots h(x)$$

By using sandwiches:

$$\lim_{X\to\infty} \frac{2-cosx}{x^2+1} = 0 \ \ldots \ldots \ldots f(x)$$

Ex (5): find $\lim_{X\to 4} g(x)\, for\ the\ function$

$$\lim_{X\to 4} |g(x) - 5| \le 3(x-4)^2$$

Sol/

$$-3(x-4)^2 \le g(x) - 5 \le 3(x-4)^2$$

$$[-3(x-4)^2 \le g(x) - 5 \le 3(x-4)^2](+5)$$

$$5 - 3(x-4)^2 \le g(x) \le 5 + 3(x-4)^2$$

$$\lim_{X\to 4} 5 - 3(x-4)^2 = 5 - 3(4-4)^2 = 5 \ldots\ldots\ldots g(x)$$

$$\lim_{X\to 4} 5 + 3(x-4)^2 = 5 + 3(4-4)^2 = 5 \ldots\ldots\ldots h(x)$$

$$\therefore \lim_{X\to 4} g(x) = 5 \ \ \ldots\ldots\ldots f(x)$$

H.W

1. $\lim_{X\to\infty} \frac{5\,x^2 + COS(7X-2)}{x^2+1}$
2. $\lim_{X\to\infty} \frac{COSX+2}{X+3}$
3. $\lim_{X\to 0} x^2 \sin\frac{1}{x^2}$
4. $\lim_{X\to 0} 3 - x^2 \cos\frac{1}{x}$

Exercises 1.2

Limit Calculations

Find the limits in Exercises 1–16.

1. $\lim_{x \to -7} (2x + 5)$
2. $\lim_{x \to 12} (10 - 3x)$
3. $\lim_{x \to 2} (-x^2 + 5x - 2)$
4. $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8)$
5. $\lim_{t \to 6} 8(t - 5)(t - 7)$
6. $\lim_{s \to 2/3} 3s(2s - 1)$
7. $\lim_{x \to 2} \frac{x + 3}{x + 6}$
8. $\lim_{x \to 5} \frac{4}{x - 7}$
9. $\lim_{y \to -5} \frac{y^2}{5 - y}$
10. $\lim_{y \to 2} \frac{y + 2}{y^2 + 5y + 6}$
11. $\lim_{x \to -1} 3(2x - 1)^2$
12. $\lim_{x \to -4} (x + 3)^{1984}$
13. $\lim_{y \to -3} (5 - y)^{4/3}$
14. $\lim_{z \to 0} (2z - 8)^{1/3}$
15. $\lim_{h \to 0} \frac{3}{\sqrt{3h + 1} + 1}$
16. $\lim_{h \to 0} \frac{5}{\sqrt{5h + 4} + 2}$

Find the limits in Exercises 17–30.

17. $\lim_{x \to 5} \frac{x - 5}{x^2 - 25}$
18. $\lim_{x \to -3} \frac{x + 3}{x^2 + 4x + 3}$
19. $\lim_{x \to -5} \frac{x^2 + 3x - 10}{x + 5}$
20. $\lim_{x \to 2} \frac{x^2 - 7x + 10}{x - 2}$
21. $\lim_{t \to 1} \frac{t^2 + t - 2}{t^2 - 1}$
22. $\lim_{t \to -1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$
23. $\lim_{x \to -2} \frac{-2x - 4}{x^3 + 2x^2}$
24. $\lim_{y \to 0} \frac{5y^3 + 8y^2}{3y^4 - 16y^2}$
25. $\lim_{u \to 1} \frac{u^4 - 1}{u^3 - 1}$
26. $\lim_{v \to 2} \frac{v^3 - 8}{v^4 - 16}$
27. $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$
28. $\lim_{x \to 4} \frac{4x - x^2}{2 - \sqrt{x}}$
29. $\lim_{x \to 1} \frac{x - 1}{\sqrt{x + 3} - 2}$
30. $\lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

ANSWER

Section 1.2, pp. 65–66

1. -9 **3.** 4 **5.** -8 **7.** 5/8 **9.** 5/2 **11.** 27 **13.** 16
15. 3/2 **17.** 1/10 **19.** -7 **21.** 3/2 **23.** $-1/2$ **25.** 4/3
27. 1/6 **29.** 4

Using the Sandwich Theorem

43. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

44. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

45. a) It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x \sin x}{2 - 2\cos x} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to 0} \frac{x \sin x}{2 - 2\cos x}?$$

ANSWER

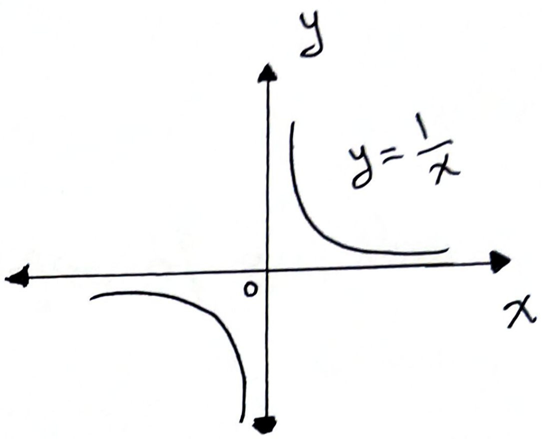
43. a) $-\infty$ b) ∞ **45.** a) ∞ b) ∞ c) ∞ d) ∞

Lecture Two :

Limits Involving Infinity :-

For example, If we have the function $y = \frac{1}{x}$

$\lim_{x \to \infty} \frac{1}{x} = 0$ and $\lim_{x \to -\infty} \frac{1}{x} = 0$



Another example is a constant function $y = k$: $\lim_{x \to \infty} k = k$

$\lim_{x \to -\infty} k = k$

Note : the limit of $\frac{\sin\theta}{\theta}$ as θ approaches $\mp\infty$ is 0

Ex :- prove that $\lim_{\theta \to \infty} \frac{\sin\theta}{\theta} = 0$

Sol :-

$$-1 \lesssim \sin\theta \lesssim 1$$

$$[\div \theta] \quad \frac{-1}{\theta} \lesssim \frac{\sin\theta}{\theta} \lesssim \frac{1}{\theta}$$

Now : $\lim_{x \to \infty} \frac{-1}{\theta} = 0$ and $\lim_{\theta \to \infty} \frac{1}{\theta} = 0$

From Sandwich theorem $\Rightarrow \lim_{\theta \to \infty} \frac{\sin\theta}{\theta} = 0$

Limits of rational functions as x approaches $\mp\infty$.

Rules :- For the rational function $\frac{f(x)}{g(x)}$

1. If degree of $f(x)$ Less than degree of $g(x)$ then $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$.

2. If degree of $f(x)$ equals degree of $g(x)$ then $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite.

3. If degree of $f(x)$ greater than degree of $g(x)$ then $\lim_{x \to \mp\infty}$ is infinite.

- To solve the limit problem of rational functions as x approaches $\mp\infty$, divide both the numerator and the denominator by the highest power of x in denominator.

Ex:-

1. $\lim_{x \to \infty} \frac{3x+1}{x^2-5} = 0$ [deg. of numerator less than deg. of denominator]

2. $\lim_{x\to\infty} \frac{-x}{7x+3} = \lim_{x\to\infty} \frac{-x/x}{(7x/x + 3/x)}$

$= \lim_{x\to\infty} \frac{-1}{7+3/x} = -1/7$

3. $\lim_{x\to\infty} \frac{-4x^3+7x}{2x^2-3x-10}$

$\lim_{x\to\infty} \frac{(-4x^3/x^2)+(7x/x^2)}{(2x^2/x^2)-(3x/x^2)-(10/x^2)}$

$\lim_{x\to\infty} \frac{-4x + 7/x}{2-3/x-10/x^2}$

$= \frac{-\infty+0}{2-0-0} = \frac{-\infty}{2} = -\infty$

4. $\lim_{x\to\infty} \frac{\cos 1/x}{1+1/x} = \frac{\cos 0}{1+0} = 1$

5. $\lim_{x\to\infty} x \sin 1/x$

$\lim_{x\to\infty} x \sin \frac{1}{x}$

let $\theta = \frac{1}{x}$

as $x \to \infty$

then $\theta \to 0$

$= \lim_{\theta\to 0} \frac{1}{\theta} \sin\theta = 1$