$Ex\ 16; Find\ \frac{dy}{dx}\ for\ the\ following\ functions:$

$$a)\ y = \cosh^{-1}(\sec x) \qquad b)\ y = \tanh^{-1}(\cos x)$$

$$c)\ y = \coth^{-1}(\sec x) \qquad d)\ y = \text{sech}^{-1}(\sin 2x)$$

$Sol:$

$$a)\frac{dy}{dx} = \frac{sec\ x \cdot \tan x}{\sqrt{\sec^2 x - 1}} = \frac{sec\ x \cdot \tan x}{\sqrt{\tan^2 x}} = \sec x \qquad where\ \tan x > 0$$

$$b)\frac{dy}{dx} = -\frac{sinx}{1 - \cos^2 x} = \frac{-\sin x}{\sin^2 x} = -\csc x$$

$$c)\frac{dy}{dx} = \frac{\sec x \cdot \tan x}{1 - \sec^2 x} = \frac{\sec x \tan x}{-\tan^2 x} = -\csc x$$

$$d)\frac{dy}{dx} = -\frac{2\cos 2x}{\sin 2x\ \cdot \sqrt{1 - \sin^2 2x}} = -2\csc 2x \quad where\ \cos 2x > 0$$

$Ex\ 17: Verify\ the\ following\ formulas:$

$$a)\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx} \qquad b)\frac{d}{dx}\tanh^{-1} u = \frac{1}{1 - u^2}\frac{du}{dx} \quad |u| < 1$$

$Proof:$

$$a)\ Let\ y = \cosh^{-1} u\ \Rightarrow u = \cosh y\ \Rightarrow \frac{du}{dx}\sinh y\frac{dy}{dx}\ \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}\frac{du}{dx}$$

$$\cosh^2 y - \sinh^2 y = 1\ \Rightarrow u^2 - \sinh^2 y = 1 \Rightarrow \sinh y = \sqrt{u^2 - 1}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}\ \Rightarrow \frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}$$

The derivatives of functions like u^v:

$Where\ u\ and\ v\ are\ differentiable\ functions\ of\ x, are\ found\ by$

$logarithmic\ differentiation:$

$$Let\ y = u^v \Rightarrow \ln y = v \cdot \ln u$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{v}{u} \cdot \frac{du}{dx} + \ln u \cdot \frac{dv}{dx}$$

$$\frac{dy}{dx} = y\left[\frac{v}{u} \cdot \frac{du}{dx} + \ln u \cdot \frac{dv}{dx}\right]$$

33) $\frac{d}{dx}u^v = u^v \cdot \left[\frac{v}{u} \cdot \frac{du}{dx} + \ln u \cdot \frac{dv}{dx}\right]$

$Ex\ 18: Find\ \frac{dy}{dx} for:$

$$a)\ y = x^{\cos x} \qquad b)\ y = (\ln x + x)^{\tan x}$$

$Sol:$

$$a)\ y = x^{\cos x} \Rightarrow \ln y = \cos x \ln x \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{x} + \ln x\ (-\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{\cos x}{x} - \sin x \ln x\right]$$

$$or\ by\ formula, where\ u = x\ and\ v = \cos x$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{x} - \sin x \ln x\right]$$

$$b) y = (\ln x + x)^{\tan x} \Rightarrow \ln y = \tan x \ln(\ln x + x)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\tan x}{\ln x + x}\left(\frac{1}{x} + 1\right) + \ln(\ln x + x)\sec^2 x$$

$$\frac{dy}{dx} = y\left[\frac{(x+1)\tan x}{x\,(\ln x + x)} + \ln(\ln x + x)\sec^2 x\right]$$

$$or\ by\ formula, where\ u = \ln x + x\ and\ v = \tan x$$

$$\frac{dy}{dx} = y\left[\frac{\tan x}{\ln x + x}\left(\frac{1}{x} + 1\right) + \ln(\ln x + x)\sec^2 x\right]$$

Applications of derivatives:

1- (L′ Hopital Rule):

$Suppose\ that\ f(x_o) = g(x_o) = 0\ and\ that\ the\ functions\ \ f\ and\ g\ \ are$ $both\ differentiable\ on\ an\ open\ interval\ (a, b)\ that\ contains\ the\ point$ $x_o. Suppose\ also\ that\ g'(x) \neq 0\ at\ every\ point\ in\ (a, b) except\ possibly$ $x_o. Then:$

$$\lim_{x\to x_o}\frac{f(x)}{g(x)} = \lim_{x\to x_o}\frac{f'(x)}{g'(x)}\ \ provided\ the\ limit\ exists.$$

$$Differentiate\ f\ and\ g\ as\ long\ as\ you\ still\ get\ the\ form \frac{0}{0}\ or \frac{\infty}{\infty}\ at\ x = x_o.$$

$Stop\ differentiating\ as\ soon\ as\ you\ get\ something\ else.$

$L'Hopital's\ rule\ does\ not\ apply\ when\ either\ the\ numerator\ or$ $denominator\ has\ a\ finite\ non - zero\ limit.$

$Ex\ 1: Evaluate\ the\ following\ limits:$

$$1)\ \lim_{x\to 0}\frac{\sin x}{x} \qquad\qquad 2)\ \lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-4}$$

$$3)\ \lim_{x\to 0}\frac{\text{x} - \sin x}{x^3} \qquad\qquad 4)\ \lim_{x\to\frac{\pi}{2}} -\left(x-\frac{\pi}{2}\right)\tan x$$

$Sol:$

1) $\lim_{x\to 0}\frac{\sin x}{x} \Rightarrow \frac{0}{0}\ using\ L'\ Hopital's\ rule\ \ \Rightarrow$
$= \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$

2) $\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-4}\ \Rightarrow \frac{0}{0}\ using\ L'\ Hopital's\ rule\ \ \Rightarrow$

$$= \lim_{x\to2} \frac{\frac{x}{\sqrt{x^2+5}}}{2x} = \lim_{x\to2} \frac{1}{2\sqrt{x^2+5}} = \frac{1}{2\sqrt{4+5}} = \frac{1}{6}$$

3) $\lim_{x\to0} \frac{x-\sin x}{x^3} \Rightarrow \frac{0}{0}$ *using L' Hopital's rule* $\Rightarrow$

$= \lim_{x\to0} \frac{1-\cos x}{3x^2} \Rightarrow \frac{0}{0}$ *using L' Hopital's rule* $\Rightarrow$

$= \lim_{x\to0} \frac{\sin x}{6x} = \frac{1}{6}$

4) $\lim_{x\to\frac{\pi}{2}^-} \left(x-\frac{\pi}{2}\right)\tan x \Rightarrow 0\cdot\infty$ *we can't using L'Hopital's rule*

$= \lim_{x\to\frac{\pi}{2}^-} \frac{x-\frac{\pi}{2}}{\cos x} \lim_{x\to\frac{\pi}{2}} \sin x \Rightarrow \frac{0}{0}$ *using L' Hopital's rule* $\Rightarrow$

$= \lim_{x\to\frac{\pi}{2}^-} \frac{1}{-\sin x} \lim_{x\to\frac{\pi}{2}} \sin x = \frac{1}{\sin\frac{\pi}{2}} \sin\frac{\pi}{2} = 1$

2 − The slope of the curve:

The derivative of the function f is the slope of the curve:

$$\text{the slope} = m = f'(x) = \frac{dy}{dx}$$

Ex 2: Write an equation for the tangent line at $x = 3$ *of the curve:*

$$f(x) = \frac{1}{\sqrt{2x+3}}$$

Sol:

$$m = f'(x) = -\frac{1}{\sqrt{(2x+3)^3}} \Rightarrow [m]_{x=3} = f'(3) = -\frac{1}{27}$$

$$f(3) = \frac{1}{\sqrt{2*3+3}} = \frac{1}{3}$$

The equation of the tangent line is :

$y - \frac{1}{3} = -\frac{1}{27}(x-3) \Rightarrow 27y + x = 12$

3 − ***Velocity and acceleration and other retes of changes:***

The average velocity of a body moving along a line is:

$$v_{av}=\frac{\Delta s}{\Delta t}=\frac{f(t+\Delta t)-f(t)}{\Delta t}=\frac{displacement}{time\ travelled}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s=f(t)$ *with respect to time* t.

i.e. $v=\frac{ds}{dt}=\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t}$

The rate at which the particle's velocity increase is called its acceleration a *. If a particle has an initial velocity* v *and a constant acceleration* a*, then its velocity after time* t *is* $v+at$.

average acceleration $=a_{av}=\frac{\Delta v}{\Delta t}$

The acceleration at an instant is $a=\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t}$

The average rate of a change in a function $y=f(x)$ *over the interval from* x *to* $x+\Delta x$ *is*:

avrage rate of change $=\frac{f(x+\Delta x)-f(x)}{\Delta x}$

The instantaneous rate of change of f *at* x *is the derivative.*

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\quad provided\ the\ limit\ exists.$$

Ex 3: *The position* s *(in meters) of a moving body as a function of time* t *(in second) is* : $s=2t^2+5t-3$; *find* :

a) *the displacement and average velocity for the time interval from* $t=0$ *to* $t=2$ *seconds.*

b) *The body's velocity at* $t=2$ *seconds.*

Sol:

a) 1) $\Delta s=s(t+\Delta t)-s(t)=2(t+\Delta t)^2+5(t+\Delta t)-3-[2t^2+5t-3]$

$$=(4t+5)\Delta t+2(\Delta t)^2$$

at $t=0$ *and* $\Delta t=2\Rightarrow \Delta s=(4*0+5)*2+2*2^2=18$

2) $v_{av}=\frac{\Delta s}{\Delta t}=\frac{(4t+5)\Delta t+2(\Delta t)^2}{\Delta t}=4t+5+2\cdot\Delta t$

at $t=0$ *and* $\Delta t=2\Rightarrow v_{av}=4*0+5+2*2=9$

b) $v(t)=\frac{d}{dt}f(t)=4t+5$

$v(2)=4*2+5=13$

$4 -$ ***Maxima and Minima***:

Increasing and decreasing function: *Let f be defined on an interval and x_1, x_2 denoted a number on that interval*:

If $f(x_1) < f(x_2)$ when ever $x_1 < x_2$ then f is increasing on that interval

If $f(x_1) > f(x_2)$ when ever $x_1 < x_2$ then f is decreasing on that interval

If $f(x_1) = f(x_2)$ for all values of x_1, x_2 then f is constant on that interval

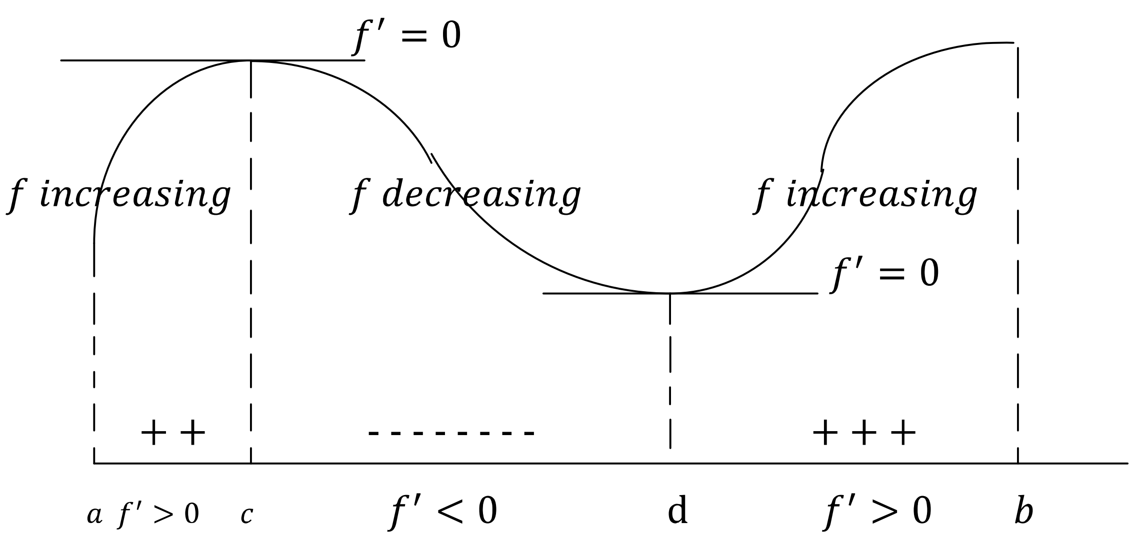
The first derivative test for rise and fall:

Suppose that a function f has a derivative at every point x of an interval I.

$$\text{then: } f \text{ increases on } I \text{ if } f'(x) > 0, \qquad \forall x \in I$$

$$f \text{ decreasing on } I \text{ if } f'(x) < 0, \qquad \forall x \in I$$

If f' changes from positive to negative values as x passes from left to right through a point c, then the value of f at c is a local maximum value of f, as shown in figer below. That is $f(c)$ is the largest value the function takes in the immediate neighborhood at $x = c$.

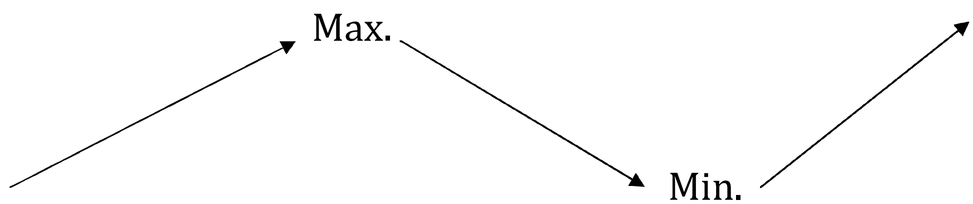


Similarly, if f' changes from negative to positive values as x passes left to right through a point d, then the value of f at d is a local minimum value of f. That is $f(d)$ is the smallest value of f takes in the immediate neighborhood of d.

Ex 5: *Graph the function* : $y = f(x) = \frac{x^3}{3} - 2x^2 + 3x + 2.$

Sol: $f'(x) = x^2 - 4x + 3 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$

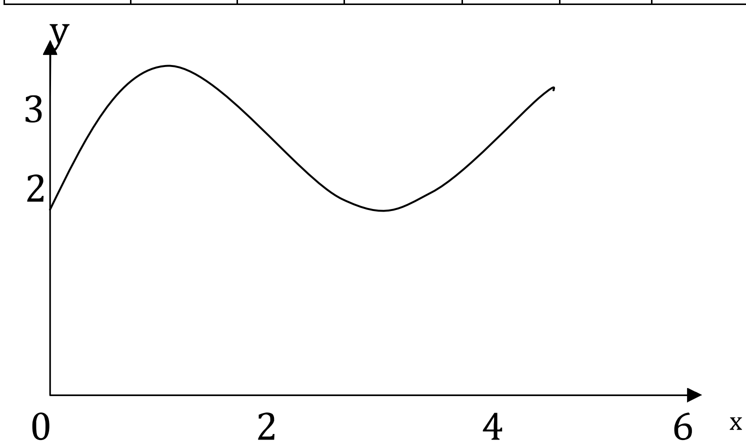
$f'(x)$ $+ + +$ 1 $- - - - -$ 3 $+ + + +$



The function has a local maximum at $x = 1$ and a local min. at $x = 3$.

To get a more accurate curve, we take :

x	0	1	2	3	4	5	6
F(x)	2	3.3	2.7	2	3.3	..	..



Concave down and concave up: The graph of a differentiable function

$y = f(x)$ *is concave down on an interval where* f' *decreases, and concave up on an interval where* f' *increases.*

The second derivative test for concavitv: the graph of $y = f(x)$ *is concave down on any interval where* $y'' < 0$, *concave up on any interval where* $y'' > 0$.

Point of inflection: A point on the curve where the concavity changes is called a point of inflection. Thus, a point of inflection on a twice − differentiable curve is a point where y'' *is positive on one side and negative on other, i.e.* $y'' = 0$.

EX-6 – Sketch the curve : $y=\frac{1}{6}(x^3-6x^2+9x+6)$.

Sol. -

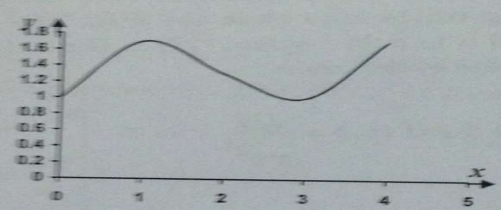
$y'=\frac{1}{2}x^2-2x+\frac{3}{2}=0 \Rightarrow x^2-4x+3=0 \Rightarrow (x-1)(x-3)=0 \Rightarrow x=1,3$

$y''=x-2 \Rightarrow$ at $x=1 \Rightarrow y''=1-2=-1<0$ concave down .

$\Rightarrow$ at $x=3 \Rightarrow y''=3-2>0$ concave up .

$\Rightarrow$ at $y''=0 \Rightarrow x-2=0 \Rightarrow x=2$ point of inflection .

x	0	1	2	3	4
y	1	1.7	1.3	1	1.7



EX-7 – What value of a makes the function :

$f(x)=x^2+\frac{a}{x}$, have :

i) a local minimum at $x=2$?
ii) a local minimum at $x=-3$?
iii) a point of inflection at $x=1$?
iv) show that the function can't have a local maximum for any value of a .

Sol. –

$f(x)=x^2+\frac{a}{x} \Rightarrow \frac{df}{dx}=2x-\frac{a}{x^2}=0 \Rightarrow a=2x^3$ and $\frac{d^2y}{dx^2}=2+\frac{2a}{x^3}$