5.7 Determination of Principal Stresses for a Member Subjected to Bi-axial Stress

When a member is subjected to bi-axial stress (*i.e.* direct stress in two mutually perpendicular planes accompanied by a simple shear stress), then the normal and shear stresses are obtained as discussed below:

Consider a rectangular body *ABCD* of uniform cross-sectional area and unit thickness subjected to normal stresses σ_1 and σ_2 as shown in Fig. 5.15 (*a*). In addition to these normal stresses, a shear stress τ also acts.

It has been shown in books on **'*Strength of Materials*'** that the normal stress across any oblique section such as *EF* inclined at an angle θ with the direction of σ_2, as shown in Fig. 5.15 (*a*), is given by

$$\sigma_t = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 + \sigma_2}{2}\cos 2\theta + \tau \sin 2\theta \quad ...(i)$$

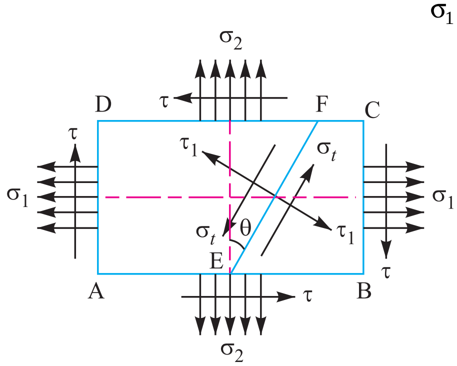
and tangential stress (*i.e.* shear stress) across the section *EF*,

$$\tau_1 = \frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta - \tau\cos 2\theta \quad ...(ii)$$

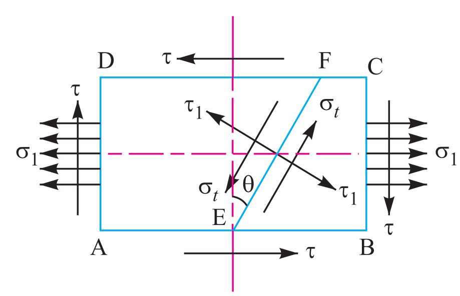
Since the planes of maximum and minimum normal stress (*i.e.* principal planes) have no shear stress, therefore the inclination of principal planes is obtained by equating $\tau_1 = 0$ in the above equation (*ii*), *i.e.*

$$\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta - \tau\cos 2\theta = 0$$

$$\therefore \quad \tan 2\theta = \frac{2\tau}{\sigma_1 - \sigma_2} \quad ...(iii)$$



(*a*) Direct stress in two mutually prependicular planes accompanied by a simple shear stress.



(*b*) Direct stress in one plane accompanied by a simple shear stress.

Fig. 5.15. Principal stresses for a member subjected to bi-axial stress.

We know that there are two principal planes at right angles to each other. Let θ_1 and θ_2 be the inclinations of these planes with the normal cross-section.

From Fig. 5.16, we find that

$$\sin 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

Contents

$$\therefore \quad \sin 2\theta_1 = + \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

and

$$\sin 2\theta_2 = - \frac{2\tau}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

Also

$$\cos 2\theta = \pm \frac{\sigma_1 - \sigma_2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

$$\therefore \quad \cos 2\theta_1 = + \frac{\sigma_1 - \sigma_2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

and

$$\cos 2\theta_2 = - \frac{\sigma_1 - \sigma_2}{\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}}$$

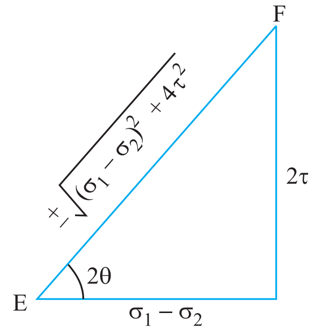


Fig. 5.16

The maximum and minimum principal stresses may now be obtained by substituting the values of sin 2θ and cos 2θ in equation ***(i)***.

∴ Maximum principal (or normal) stress,

$$\sigma_{t1} = \frac{\sigma_1 + \sigma_2}{2} + \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \quad ...(iv)$$

and minimum principal (or normal) stress,

$$\sigma_{t2} = \frac{\sigma_1 + \sigma_2}{2} - \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \quad ...(v)$$

The planes of maximum shear stress are at right angles to each other and are inclined at 45° to the principal planes. The maximum shear stress is given by ***one-half the algebraic difference between the principal stresses,*** *i.e.*

$$\tau_{max} = \frac{\sigma_{t1} - \sigma_{t2}}{2} = \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2} \quad ...(vi)$$

A Boring mill.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Top

Contents

Notes: 1. When a member is subjected to direct stress in one plane accompanied by a simple shear stress as shown in Fig. 5.15 (*b*), then the principal stresses are obtained by substituting $\sigma_2 = 0$ in equation **(*iv*), (*v*)** and **(*vi*).**

$$\therefore \qquad \sigma_{t1} = \frac{\sigma_1}{2} + \frac{1}{2}\left[\sqrt{(\sigma_1)^2 + 4\,\tau^2}\right]$$

$$\sigma_{t2} = \frac{\sigma_1}{2} - \frac{1}{2}\left[\sqrt{(\sigma_1)^2 + 4\,\tau^2}\right]$$

and

$$\tau_{max} = \frac{1}{2}\left[\sqrt{(\sigma_1)^2 + 4\,\tau^2}\right]$$

2. In the above expression of σ_{t2}, the value of $\frac{1}{2}\left[\sqrt{(\sigma_1)^2 + 4\,\tau^2}\right]$ is more than $\frac{\sigma_1}{2}$. Therefore the nature of σ_{t2} will be opposite to that of σ_{t1}, *i.e.* if σ_{t1} is tensile then σ_{t2} will be compressive and *vice-versa*.

5.8 Application of Principal Stresses in Designing Machine Members

There are many cases in practice, in which machine members are subjected to combined stresses due to simultaneous action of either tensile or compressive stresses combined with shear stresses. In many shafts such as propeller shafts, C-frames etc., there are direct tensile or compressive stresses due to the external force and shear stress due to torsion, which acts normal to direct tensile or compressive stresses. The shafts like crank shafts, are subjected simultaneously to torsion and bending. In such cases, the maximum principal stresses, due to the combination of tensile or compressive stresses with shear stresses may be obtained.

The results obtained in the previous article may be written as follows:

1. Maximum tensile stress,

$$\sigma_{t(max)} = \frac{\sigma_t}{2} + \frac{1}{2}\left[\sqrt{(\sigma_t)^2 + 4\,\tau^2}\right]$$

2. Maximum compressive stress,

$$\sigma_{c(max)} = \frac{\sigma_c}{2} + \frac{1}{2}\left[\sqrt{(\sigma_c)^2 + 4\,\tau^2}\right]$$

3. Maximum shear stress,

$$\tau_{max} = \frac{1}{2}\left[\sqrt{(\sigma_t)^2 + 4\,\tau^2}\right]$$

where σ_t = Tensile stress due to direct load and bending,

σ_c = Compressive stress, and

τ = Shear stress due to torsion.

Notes : 1. When $\tau = 0$ as in the case of thin cylindrical shell subjected in internal fluid pressure, then

$$\sigma_{t\,(max)} = \sigma_t$$

2. When the shaft is subjected to an axial load (P) in addition to bending and twisting moments as in the propeller shafts of ship and shafts for driving worm gears, then the stress due to axial load must be added to the bending stress (σ_b). This will give the resultant tensile stress or compressive stress (σ_t or σ_c) depending upon the type of axial load (*i.e.* pull or push).

Example 5.13. *A hollow shaft of 40 mm outer diameter and 25 mm inner diameter is subjected to a twisting moment of 120 N-m, simultaneously, it is subjected to an axial thrust of 10 kN and a bending moment of 80 N-m. Calculate the maximum compressive and shear stresses.*

Solution. Given: d_o = 40 mm ; d_i = 25 mm ; T = 120 N-m = 120 × 10^3 N-mm ; P = 10 kN = 10 × 10^3 N ; M = 80 N-m = 80 × 10^3 N-mm

We know that cross-sectional area of the shaft,

$$A = \frac{\pi}{4}\left[(d_o)^2 - (d_i)^2\right] = \frac{\pi}{4}\left[(40)^2 - (25)^2\right] = 766 \text{ mm}^2$$

Top

∴ Direct compressive stress due to axial thrust,

$$\sigma_o = \frac{P}{A} = \frac{10 \times 10^3}{766} = 13.05 \text{ N/mm}^2 = 13.05 \text{ MPa}$$

Section modulus of the shaft,

$$Z = \frac{\pi}{32}\left[\frac{(d_o)^4 - (d_i)^4}{d_o}\right] = \frac{\pi}{32}\left[\frac{(40)^4 - (25)^4}{40}\right] = 5325 \text{ mm}^3$$

∴ Bending stress due to bending moment,

$$\sigma_b = \frac{M}{Z} = \frac{80 \times 10^3}{5325} = 15.02 \text{ N/mm}^2 = 15.02 \text{ MPa (compressive)}$$

and resultant compressive stress,

$$\sigma_c = \sigma_b + \sigma_o = 15.02 + 13.05 = 28.07 \text{ N/mm}^2 = 28.07 \text{ MPa}$$

We know that twisting moment (T),

$$120 \times 10^3 = \frac{\pi}{16} \times \tau \left[\frac{(d_o)^4 - (d_i)^4}{d_o}\right] = \frac{\pi}{16} \times \tau \left[\frac{(40)^4 - (25)^4}{40}\right] = 10\ 650\ \tau$$

$$\therefore \quad \tau = 120 \times 10^3/10\ 650 = 11.27 \text{ N/mm}^2 = 11.27 \text{ MPa}$$

Maximum compressive stress

We know that maximum compressive stress,

$$\sigma_{c(max)} = \frac{\sigma_c}{2} + \frac{1}{2}\left[\sqrt{(\sigma_c)^2 + 4\,\tau^2}\right]$$

$$= \frac{28.07}{2} + \frac{1}{2}\left[\sqrt{(28.07)^2 + 4\,(11.27)^2}\right]$$

$$= 14.035 + 18 = 32.035 \text{ MPa}$$ **Ans.**

Maximum shear stress

We know that maximum shear stress,

$$\tau_{max} = \tfrac{1}{2}\left[\sqrt{(\sigma_c)^2 + 4\tau^2}\right] = \tfrac{1}{2}\left[\sqrt{(28.07)^2 + 4(11.27)^2}\right] = 18 \text{ MPa}$$ **Ans.**

Example 5.14. *A shaft, as shown in Fig. 5.17, is subjected to a bending load of 3 kN, pure torque of 1000 N-m and an axial pulling force of 15 kN.*

Calculate the stresses at A and B.

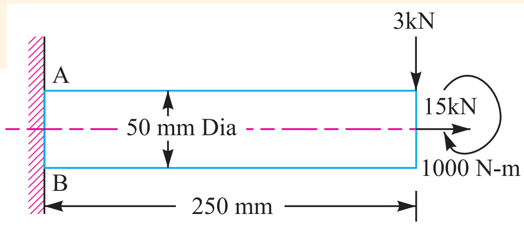


Fig. 5.17

Solution. Given : W = 3 kN = 3000 N ; T = 1000 N-m = 1×10^6 N-mm ; P = 15 kN = 15×10^3 N ; d = 50 mm; x = 250 mm

We know that cross-sectional area of the shaft,

$$A = \frac{\pi}{4} \times d^2$$

$$= \frac{\pi}{4}(50)^2 = 1964 \text{ mm}^2$$

∴ Tensile stress due to axial pulling at points A and B,

$$\sigma_o = \frac{P}{A} = \frac{15 \times 10^3}{1964} = 7.64 \text{ N/mm}^2 = 7.64 \text{ MPa}$$

Bending moment at points A and B,

$$M = W.x = 3000 \times 250 = 750 \times 10^3 \text{ N-mm}$$

Section modulus for the shaft,

$$Z = \frac{\pi}{32} \times d^3 = \frac{\pi}{32} (50)^3$$

$$= 12.27 \times 10^3 \text{ mm}^3$$

∴ Bending stress at points *A* and *B*,

$$\sigma_b = \frac{M}{Z} = \frac{750 \times 10^3}{12.27 \times 10^3}$$

$$= 61.1 \text{ N/mm}^2 = 61.1 \text{ MPa}$$

This picture shows a machine component inside a crane

Note : This picture is given as additional information and is not a direct example of the current chapter.

This bending stress is tensile at point *A* and compressive at point *B*.

∴ Resultant tensile stress at point *A*,

$$\sigma_A = \sigma_b + \sigma_o = 61.1 + 7.64$$
$$= 68.74 \text{ MPa}$$

and resultant compressive stress at point *B*,

$$\sigma_B = \sigma_b - \sigma_o = 61.1 - 7.64 = 53.46 \text{ MPa}$$

We know that the shear stress at points *A* and *B* due to the torque transmitted,

$$\tau = \frac{16\,T}{\pi\, d^3} = \frac{16 \times 1 \times 10^6}{\pi\,(50)^3} = 40.74 \text{ N/mm}^2 = 40.74 \text{ MPa} \qquad \ldots \left(\because\ T = \frac{\pi}{16} \times \tau \times d^3\right)$$

Stresses at point A

We know that maximum principal (or normal) stress at point *A*,

$$\sigma_{A(max)} = \frac{\sigma_A}{2} + \frac{1}{2}\left[\sqrt{(\sigma_A)^2 + 4\,\tau^2}\right]$$

$$= \frac{68.74}{2} + \frac{1}{2}\left[\sqrt{(68.74)^2 + 4\,(40.74)^2}\right]$$

$$= 34.37 + 53.3 = 87.67 \text{ MPa (tensile)}$$ **Ans.**

Minimum principal (or normal) stress at point *A*,

$$\sigma_{A(min)} = \frac{\sigma_A}{2} - \frac{1}{2}\left[\sqrt{(\sigma_A)^2 + 4\,\tau^2}\right] = 34.37 - 53.3 = -18.93 \text{ MPa}$$

$$= 18.93 \text{ MPa (compressive)}$$ **Ans.**

and maximum shear stress at point *A*,

$$\tau_{A(max)} = \frac{1}{2}\left[\sqrt{(\sigma_A)^2 + 4\,\tau^2}\right] = \frac{1}{2}\left[\sqrt{(68.74)^2 + 4\,(40.74)^2}\right]$$

$$= 53.3 \text{ MPa}$$ **Ans.**

Stresses at point B

We know that maximum principal (or normal) stress at point *B*,

$$\sigma_{B(max)} = \frac{\sigma_B}{2} + \frac{1}{2}\left[\sqrt{(\sigma_B)^2 + 4\,\tau^2}\right]$$

$$= \frac{53.46}{2} + \frac{1}{2}\left[\sqrt{(53.46)^2 + 4\,(40.74)^2}\right]$$

$$= 26.73 + 48.73 = 75.46 \text{ MPa (compressive)}$$ **Ans.**

Minimum principal (or normal) stress at point *B*,

$$\sigma_{B(min)} = \frac{\sigma_B}{2} - \frac{1}{2}\left[\sqrt{(\sigma_B)^2 + 4\tau^2}\right]$$

$$= 26.73 - 48.73 = -22 \text{ MPa}$$

$$= 22 \text{ MPa (tensile)}$$ **Ans.**

and maximum shear stress at point *B*,

$$\tau_{B(max)} = \frac{1}{2}\left[\sqrt{(\sigma_B)^2 + 4\,\tau^2}\right] = \frac{1}{2}\left[\sqrt{(53.46)^2 + 4\,(40.74)^2}\right]$$

$$= 48.73 \text{ MPa}$$ **Ans.**

Example 5.15. *An overhang crank with pin and shaft is shown in Fig. 5.18. A tangential load of 15 kN acts on the crank pin. Determine the maximum principal stress and the maximum shear stress at the centre of the crankshaft bearing.*

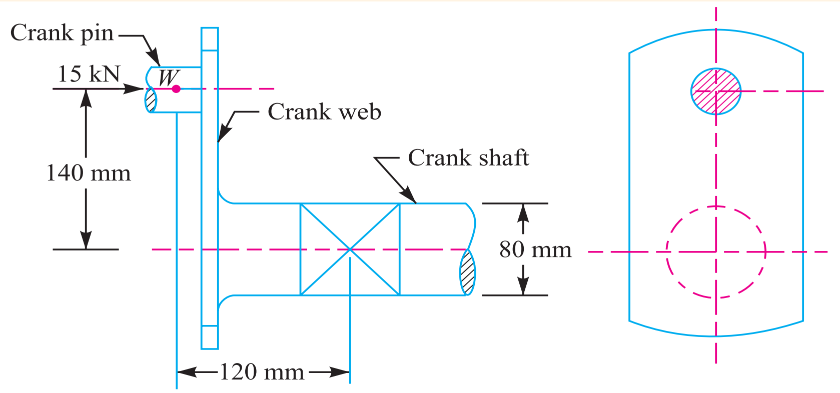


Fig. 5.18

Solution. Given : $W = 15 \text{ kN} = 15 \times 10^3$ N ; $d = 80$ mm ; $y = 140$ mm ; $x = 120$ mm

Bending moment at the centre of the crankshaft bearing,

$$M = W \times x = 15 \times 10^3 \times 120 = 1.8 \times 10^6 \text{ N-mm}$$

and torque transmitted at the axis of the shaft,

$$T = W \times y = 15 \times 10^3 \times 140 = 2.1 \times 10^6 \text{ N-mm}$$

We know that bending stress due to the bending moment,

$$\sigma_b = \frac{M}{Z} = \frac{32\,M}{\pi\,d^3} \qquad \ldots\left(\because Z = \frac{\pi}{32} \times d^3\right)$$

$$= \frac{32 \times 1.8 \times 10^6}{\pi\,(80)^3} = 35.8 \text{ N/mm}^2 = 35.8 \text{ MPa}$$

and shear stress due to the torque transmitted,

$$\tau = \frac{16\,T}{\pi\,d^3} = \frac{16 \times 2.1 \times 10^6}{\pi\,(80)^3} = 20.9 \text{ N/mm}^2 = 20.9 \text{ MPa}$$

Maximum principal stress

We know that maximum principal stress,

$$\sigma_{t(max)} = \frac{\sigma_t}{2} + \frac{1}{2}\left[\sqrt{(\sigma_t)^2 + 4\,\tau^2}\right]$$

$$= \frac{35.8}{2} + \frac{1}{2}\left[\sqrt{(35.8)^2 + 4\,(20.9)^2}\right] \qquad \ldots (\text{Substituting } \sigma_t = \sigma_b)$$

$$= 17.9 + 27.5 = 45.4 \text{ MPa}$$ **Ans.**

Maximum shear stress

We know that maximum shear stress,

$$\tau_{max} = \frac{1}{2}\left[\sqrt{(\sigma_t)^2 + 4\,\tau^2}\right] = \frac{1}{2}\left[\sqrt{(35.8)^2 + 4\,(20.9)^2}\right]$$

$$= 27.5 \text{ MPa}$$ **Ans.**

5.9 Theories of Failure Under Static Load

It has already been discussed in the previous chapter that strength of machine members is based upon the mechanical properties of the materials used. Since these properties are usually determined from simple tension or compression tests, therefore, predicting failure in members subjected to uni-axial stress is both simple and straight-forward. But the problem of predicting the failure stresses for members subjected to bi-axial or tri-axial stresses is much more complicated. In fact, the problem is so complicated that a large number of different theories have been formulated. The principal theories of failure for a member subjected to bi-axial stress are as follows:

1. Maximum principal (or normal) stress theory (also known as Rankine's theory).

2. Maximum shear stress theory (also known as Guest's or Tresca's theory).

3. Maximum principal (or normal) strain theory (also known as Saint Venant theory).

4. Maximum strain energy theory (also known as Haigh's theory).

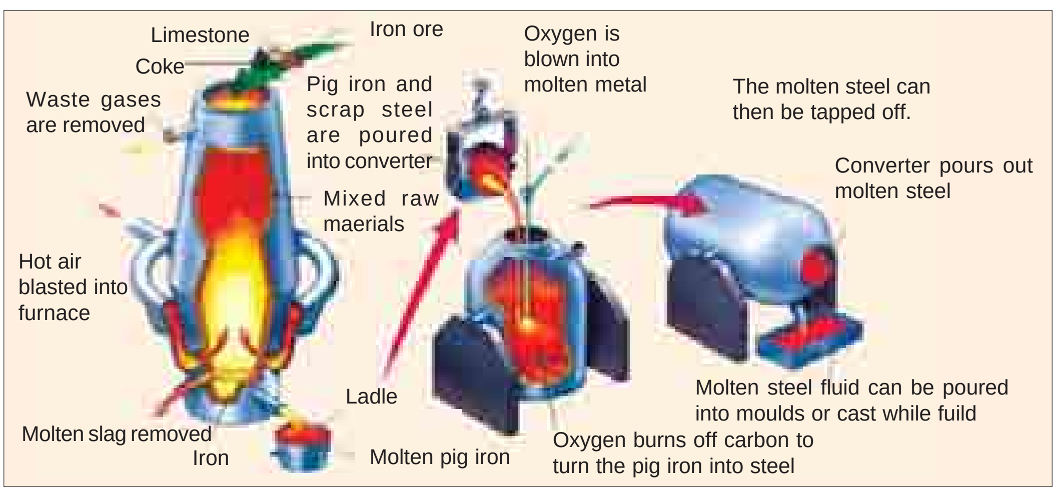
5. Maximum distortion energy theory (also known as Hencky and Von Mises theory).

Since ductile materials usually fail by yielding *i.e.* when permanent deformations occur in the material and brittle materials fail by fracture, therefore the limiting strength for these two classes of materials is normally measured by different mechanical properties. For ductile materials, the limiting strength is the stress at yield point as determined from simple tension test and it is, assumed to be equal in tension or compression. For brittle materials, the limiting strength is the ultimate stress in tension or compression.

5.10 Maximum Principal or Normal Stress Theory (Rankine's Theory)

According to this theory, the failure or yielding occurs at a point in a member when the maximum principal or normal stress in a bi-axial stress system reaches the limiting strength of the material in a simple tension test.

Since the limiting strength for ductile materials is yield point stress and for brittle materials (which do not have well defined yield point) the limiting strength is ultimate stress, therefore according



Pig iron is made from iron ore in a blast furnace. It is a brittle form of iron that contains 4-5 per cent carbon.

Note : This picture is given as additional information and is not a direct example of the current chapter.