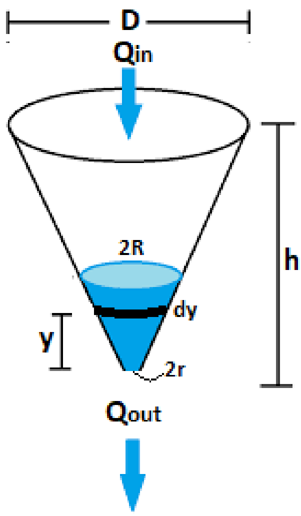
Applications of First Order Differential Equations

أولا: حساب التسرب من الخزانات

إذا كان لدينا خزان مملوء بالماء وهذا الخزان يحتوي على ثقب فان الماء يبدأ بالتسرب خلال الزمن وبمرور الزمن يبدأ عمق الماء بالنقصان، لذلك حساب مقدار ارتفاع الماء داخل الخزان المثقوب يتغير مع الزمن ويحسب من خلال المعادلة:

$$A(y) \cdot \frac{dy}{dt} = Q_{in} - Q_{out}$$



حيث:

$A(y)$: مساحة الشريحة عند العمق y.

Q_{in}: التصريف الداخل الى الخزان.

Q_{out}: التصريف الخارج من الخزان ويحسب من خلال المعادلة التالية:

$$Q_{out} = \pi\, r^2 \sqrt{2gy}$$

حيث:

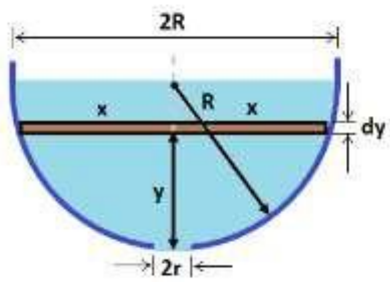
r: نصف قطر الفتحة التي يتسرب منها الماء.

g: التعجيل الأرضي ويساوي $9.81\ m/sec^2$.

y: بعد الشريحة من الفتحة (مقدار تغير ارتفاع الماء مع الزمن)

Example (1): A hemispherical tank of radius $(R = 3\ m)$ in initially filled with water. At the bottom of the tank, there is a hole of radius $(r = 1\ cm)$ through which the water drains under the influence of gravity. Find the depth of the water in the tank at any time t, and determine how long it will take the tank to drain completely.

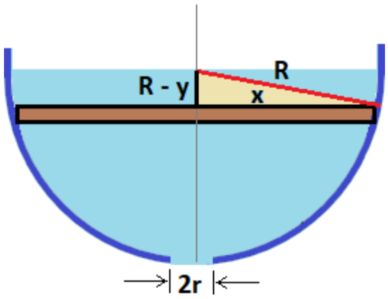
Solve:



$$A(y) \cdot \frac{dy}{dt} = Q_{in} - Q_{out}$$

$$A(y) = \pi \cdot x^2$$

From figure of tank:



$x^2 + (R - y)^2 = R^2$

$x^2 = R^2 - (R^2 - 2Ry + y^2)$

$x^2 = 2Ry - y^2 \quad \rightarrow \quad x^2 = 6y - y^2 \; sub \; in \; eq.(1)$

$\therefore A(y) = \pi \cdot (6y - y^2)$

$Q_{in} = 0$

$Q_{out} = \pi \, r^2 \sqrt{2gy} = \pi \, (0.01)^2 \sqrt{2 \times 9.81} \sqrt{y} = 4.429 \times 10^{-4} \, \pi \, y^{1/2}$

$\therefore \pi \cdot (6y - y^2) \cdot \dfrac{dy}{dt} = 0 - 4.429 \times 10^{-4} \, \pi \, y^{1/2} \qquad re - arrengement$

$(6y - y^2) \cdot dy = -\, 4.429 \times 10^{-4} \; y^{1/2} \cdot dt$

$\dfrac{(6y - y^2)}{y^{1/2}} \cdot dy = -4.429 \times 10^{-4} \cdot dt$

$y^{-1/2}(6y - y^2) \cdot dy = -\, 4.429 \times 10^{-4} \cdot dt$

$\int (6y^{1/2} - y^{3/2}) \cdot dy = \int -\, 4.429 \times 10^{-4} \cdot dt$

$6 \dfrac{y^{3/2}}{3/2} - \dfrac{y^{5/2}}{5/2} = -\, 4.429 \times 10^{-4} \, t + c$

$4 \, y^{3/2} - \dfrac{2}{5} \, y^{5/2} = -4.429 \times 10^{-4} \, t + c$

Apply boundary condition: $at \; t = 0 \qquad y = h = R$ (total height of water) $= 3 \, m$

$4 \times (3)^{3/2} - \dfrac{2}{5}(3)^{5/2} = -\, 4.429 \times 10^{-4} \; \times 0 + c \qquad \rightarrow$

$c = 14.549$

$\therefore 4 \, y^{3/2} - \dfrac{2}{5} \, y^{5/2} = -\, 4.429 \times 10^{-4} \, t + 14.549 \qquad at \; any \; time$

The time to empty the tank at $y = 0$:

$\therefore 4 \, (0)^{3/2} - \dfrac{2}{5} \, (0)^{5/2} = -\, 4.429 \times 10^{-4} \, t + 14.549 \qquad \rightarrow$

$t = 32849.9 \; sec$

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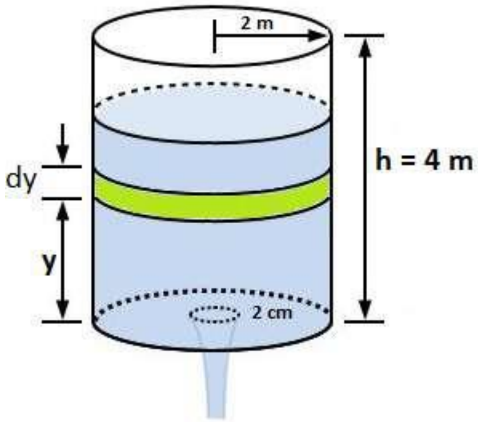
Example (2): **A cylindrical tank with radius (2m) and height of (4m) has initially filled with water. In the bottom of the tank, there is a hole of diameter (2 cm) through which the water drains under the influence of gravity. Find:**

1. **The depth of the water in the tank at any time t.**
2. **The time to reach the height of water 1 m.**
3. **The time to empty the tank.**

Solve:



1.

$$A(y) \cdot \frac{dy}{dt} = Q_{in} - Q_{out}$$

$A(y) = \pi \cdot 2^2$ مساحة الشريحة ثابتة

$$Q_{in} = 0$$

$$Q_{out} = \pi\, r^2 \sqrt{2gy}$$

$$r = \frac{d}{2} = \frac{2 \times 10^{-2}}{2} = 0.01\, m$$

$$\therefore \pi \cdot 2^2 \cdot \frac{dy}{dt} = 0 - \pi\, 0.01^2 \sqrt{2 \times 9.81\, y}$$

$$4\, dy = -\, 4.427 \times 10^{-4} \sqrt{y} \cdot dt \qquad re - arrengement$$

$$\frac{4}{\sqrt{y}} dy = -\, 4.427 \times 10^{-4} \cdot dt$$

$$\int 4\, y^{-1/2}\, dy = \int -\, 4.427 \times 10^{-4} \cdot dt$$

$$4 \frac{y^{1/2}}{1/2} = -\, 4.427 \times 10^{-4}\, t + c$$

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2. The time to reach the height of water 1 m.

$at\ y = 1\ m \quad \rightarrow \qquad 8\sqrt{1} = -\,4.427 \times 10^{-4}\ t + 16 \quad \rightarrow$

$t = 18070.9\ sec$

3. The time to empty the tank.

The time to empty the tank at $y = 0$:

$8\sqrt{0} = -\,4.427 \times 10^{-4}\ t + 16 \qquad \rightarrow \qquad t = 36141.8\ sec$

H.W: A conical tank with diameter (3 m) from top and (5 m) depth, initially filled with water. At the bottom of the tank, there is a hole of radius (0.02 m). Find the depth of the water in the tank at any time t, and how long it will take the tank to empty.

Ans: $0.036\ y^{2.5} = -1.772 \times 10^{-3} t + 2.012,\ \ t = 1135.44\ sec$

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