Chapter six

Methods of integration

6-1- Integration by parts:

The formula for integration by parts comes from the product rule:-

$$d(u\cdot v) = u\cdot dv + v\cdot du \quad \Rightarrow \quad u\cdot dv = d(u\cdot v) - v\cdot du$$

and integrated to give: $\int u\,dv = \int d(u\cdot v) - \int v\,du$

then the integration by parts formula is:-

$$\int u\,dv = u\cdot v - \int v\,du$$

Rule for choosing u and dv is:

For u: choose something that becomes simpler when differentiated.

For dv: choose something whose integral is simple.

It is not always possible to follow this rule, but when we can.

EX-1 **– Evaluate the following integrals:**

1) $\int xe^x\,dx$

2) $\int x\cdot cosx\,dx$

3) $\int \frac{x}{\sqrt{x-1}}\,dx$

4) $\int x^2\cdot ln\,x\,dx$

5) $\int x\cdot sec^2x\,dx$

6) $\int ln\left(x+\sqrt{1+x^2}\right)dx$

7) $\int sin^{-1}ax\,dx$

8) $\int e^{ax}\cdot sinbx\,dx$

9) $\int x^3\cdot e^x\,dx$

10) $\int x^3\cdot e^{x^2}\,dx$

Sol. **–**

1) let $\left.\begin{array}{l} u = x \Rightarrow du = dx \\ dv = e^x dx \Rightarrow v = e^x \end{array}\right\} \Rightarrow \int udv = u\cdot v - \int vdu$

$$\int x\cdot e^x dx = x\cdot e^x - \int e^x dx = x\cdot e^x - e^x + c$$

$$2)\quad let\quad \left.\begin{array}{l} u = x \quad \Rightarrow \quad du = dx \\ dv = \cos x\, dx \ \Rightarrow\ v = \sin x \end{array}\right\} \Rightarrow \quad \int u\,dv = u \cdot v - \int v\,du$$

$$\int x \cdot \cos x\, dx = x \cdot \sin x - \int \sin x\, dx = x \cdot \sin x + \cos x + c$$

$$3)\quad let\quad \left.\begin{array}{l} u = x \quad \Rightarrow \quad du = dx \\ dv = \dfrac{1}{\sqrt{x-1}}dx \ \Rightarrow\ v = 2(x-1)^{1/2} \end{array}\right\} \Rightarrow \quad \int u\,dv = u \cdot v - \int v\,du$$

$$\int \frac{x}{\sqrt{x-1}}dx = 2x \cdot (x-1)^{1/2} - 2\int (x-1)^{1/2} dx$$

$$= 2x \cdot \sqrt{x-1} - \frac{2(x-1)^{3/2}}{3/2} + c = 2x \cdot \sqrt{x-1} - \frac{4}{3}\sqrt{(x-1)^3} + c$$

$$4)\quad let\quad \left.\begin{array}{l} u = \ln x \quad \Rightarrow \quad du = \dfrac{1}{x}dx \\ dv = x^2 dx \ \Rightarrow\ v = \dfrac{x^3}{3} \end{array}\right\} \Rightarrow \quad \int u\,dv = u \cdot v - \int v\,du$$

$$\int x^2 \cdot \ln x\, dx = \frac{x^3}{3} \cdot \ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3} \cdot \ln x - \frac{1}{9}x^3 + c$$

$$5)\quad let\quad \left.\begin{array}{l} u = x \quad \Rightarrow \quad du = dx \\ dv = \sec^2 x\, dx \ \Rightarrow\ v = \tan x \end{array}\right\} \Rightarrow \quad \int u\,dv = u \cdot v - \int v\,du$$

$$\int x \cdot \sec^2 x\, dx = x \cdot \tan x - \int \tan x\, dx = x \cdot \tan x + \ln|\cos x| + c$$

$$6)\quad let\ \ u = \ln\left(x + \sqrt{1+x^2}\right) \ \Rightarrow\ du = \frac{1 + \dfrac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}}dx$$

$$dv = dx \quad \Rightarrow \quad v = x$$

$$\int \ln\left(x + \sqrt{1+x^2}\right)dx = x \cdot \ln\left(x + \sqrt{1+x^2}\right) - \int x\left(1+x^2\right)^{-1/2} dx$$

$$= x \cdot \ln\left(x + \sqrt{1+x^2}\right) - \frac{1}{2} \cdot \frac{\left(1+x^2\right)^{1/2}}{1/2} + c = x \cdot \ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + c$$

7) let $u = \sin^{-1} ax \Rightarrow du = \dfrac{a\,dx}{\sqrt{1-a^2x^2}}$ & $dv = dx \Rightarrow v = x$

$$\int \sin^{-1} ax\,dx = x\cdot \sin^{-1} ax - \int \frac{a\,x}{\sqrt{1-a^2x^2}}dx$$

$$= x\cdot \sin^{-1} ax + \frac{1}{2a}\int -2a^2x\left(1-a^2x^2\right)^{-1/2}dx$$

$$= x\cdot \sin^{-1} ax + \frac{1}{2a}\cdot\frac{\left(1-a^2x^2\right)^{1/2}}{1/2} + c = x\cdot \sin^{-1} ax + \frac{\sqrt{1-a^2x^2}}{a} + c$$

8) let $u = e^{ax} \Rightarrow du = a\cdot e^{ax}\,dx$ & $dv = \sin bx\,dx \Rightarrow v = -\dfrac{1}{b}\cos bx$

$$\int e^{ax}\cdot \sin bx\,dx = -\frac{1}{b}e^{ax}\cdot\cos bx + \frac{a}{b}\int e^{ax}\cdot\cos bx\,dx \quad ....(1)$$

let $u = e^{ax} \Rightarrow du = a\cdot e^{ax}\,dx$ & $dv = \cos bx\,dx \Rightarrow v = \dfrac{1}{b}\sin bx$

$$\int e^{ax}\cdot \cos bx\,dx = \frac{1}{b}e^{ax}\cdot\sin bx - \frac{a}{b}\int e^{ax}\cdot\sin bx\,dx \quad ....(2)$$

sub. (2) in (1) $\Rightarrow$

$$\int e^{ax}\cdot \sin bx\,dx = -\frac{1}{b}e^{ax}\cdot\cos bx + \frac{a}{b^2}e^{ax}\cdot\sin bx\,dx - \frac{a^2}{b^2}\int e^{ax}\cdot\sin bx\,dx$$

$$\int e^{ax}\cdot \sin bx\,dx + \frac{a^2}{b^2}\int e^{ax}\cdot\sin bx\,dx = -\frac{1}{b}e^{ax}\cdot\cos bx + \frac{a}{b^2}e^{ax}\cdot\sin bx\,dx + c$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cdot \sin bx\,dx = \frac{e^{ax}}{b^2}\left(a\sin bx - b\cos bx\right) + c$$

$$\therefore \quad \int e^{ax}\cdot \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) + c$$

9) <u>derivative of u</u> <u>integration of dv</u>

derivative of u		integration of dv
x^3	+	e^x
$3x^2$	-	e^x
$6x$	+	e^x
6	-	e^x
0		e^x

$$\therefore \int x^3 e^{ax}\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c$$

$$= e^x\left(x^3 - 3x^2 + 6x - 6\right) + c$$

$$10)\quad let \quad u = x^2 \;\Rightarrow\; du = 2x\,dx \quad \& \quad dv = x \cdot e^{x^2} dx \;\Rightarrow\; v = \frac{1}{2}e^{x^2}$$

$$\int x^3 \cdot e^{x^2} dx = \frac{1}{2}x^2 \cdot e^{x^2} - \frac{1}{2}\int 2x \cdot e^{x^2} dx = \frac{1}{2}x^2 \cdot e^{x^2} - \frac{1}{2}e^{x^2} + c$$

6-2- Odd and even powers of sine and cosine:

To integrate an odd positive power of $sinx$ (say $sin^{2n+1}x$) we split off a factor of $sinx$ and rewrite the remaining even power in terms of the cosine. We write:-

$$\int sin^{2n+1} x \cdot dx = \int \left(1 - cos^2 x\right)^n \cdot sin\,x \; dx$$

$$and \int cos^{2n+1} x \cdot dx = \int \left(1 - sin^2 x\right)^n \cdot cos\,x \; dx$$

***EX-2*- Evaluate:**

$1) \int sin^3 x\, dx$ $\qquad$ $2) \int cos^5 x\, dx$

***Sol.*-**

$$1) \int sin^3 x\, dx = \int sin^2 x \cdot sinx\, dx = \int \left(1 - cos^2 x\right) \cdot sinx\, dx$$

$$= \int sinx\, dx + \int cos^2 x \cdot \left(-sinx\right) dx = -cos\,x + \frac{1}{3}cos^3 x + c$$

$$2) \int cos^5 x\, dx = \int cos^4 x \cdot cosx \; dx = \int \left(1 - sin^2 x\right)^2 \cdot cos\,x\, dx$$

$$= \int cosx\, dx - 2\int sin^2 x \cdot cos\,x\, dx + \int sin^4 x \cdot cos\,x\, dx$$

$$= sinx - \frac{2}{3}sin^3 x + \frac{1}{5}sin^5 x + c$$

To integrate an even positive power of sine (say $sin^{2n}x$) we use the relations:-

$$cos^2\theta = \frac{1 + cos\,2\theta}{2} \qquad or \qquad sin^2\theta = \frac{1 - cos\,2\theta}{2}$$

then we can write:-

$$\int \sin^{2n} x \cdot dx = \int \left(\frac{1-\cos 2x}{2} \right)^n \ dx$$

$$\text{and } \int \cos^{2n} x \cdot dx = \int \left(\frac{1+\cos 2x}{2} \right)^n \ dx$$

***EX-3*- Evaluate:**

1) $\int \cos^2 \theta \, d\theta$ 2) $\int \sin^4 \theta \, d\theta$

***Sol.*-**

$$1)\ \int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\left[\int d\theta + \frac{1}{2}\int 2\cos 2\theta \, d\theta \right]$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta \right] + c$$

$$2)\ \int \sin^4\theta \, d\theta = \int \left(\frac{1-\cos 2\theta}{2} \right)^2 d\theta = \frac{1}{4}\left[\int d\theta - \int \cos 2\theta (2d\theta) + \int \cos^2 2\theta \, d\theta \right]$$

$$= \frac{1}{4}\left[\theta - \sin 2\theta + \int \frac{1+\cos 4\theta}{2} d\theta \right] = \frac{1}{4}\left[\theta - \sin 2\theta + \frac{1}{2}(\theta + \frac{1}{4}\sin 4\theta) \right] + c$$

$$= \frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + c$$

To integrate the following identities:-

$$\int \sin mx \cdot \sin nx \ dx \ , \int \sin mx \cdot \cos nx \ dx \ ,\text{and } \int \cos mx \cdot \cos nx \ dx$$

we use the following formulas:-

$$\sin mx \cdot \sin nx = \frac{\cos(m-n)x - \cos(m+n)x}{2}$$

$$\sin mx \cdot \cos nx = \frac{\sin(m-n)x + \sin(m+n)x}{2}$$

$$\cos mx \cdot \cos nx = \frac{\cos(m-n)x + \cos(m+n)x}{2}$$

EX-4- **Evaluate:**

$$1)\ \int \sin 3x \cdot \cos 5x\, dx \qquad 2)\ \int \cos x \cdot \cos 7x\, dx \qquad 3)\int \sin x \cdot \sin 2x\, dx$$

Sol.-

$$1)\ \int \sin 3x \cdot \cos 5x\, dx = \frac{1}{2}\int \big(\sin(3x-5x) + \sin(3x+5x)\big)dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}\int \sin 2x(2dx) + \frac{1}{8}\int \sin 8x(8dx)\right] = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + c$$

$$2)\ \int \cos x \cdot \cos 7x\, dx = \frac{1}{2}\int \big(\cos(6x) + \cos(8x)\big)dx = \frac{1}{12}\sin 6x + \frac{1}{16}\sin 8x + c$$

$$3)\ \int \sin x \cdot \sin 2x\, dx = \frac{1}{2}\int \big(\cos x - \cos 3x\big)dx = \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + c$$

6-3- Trigonometric substitutions:

Trigonometric substitutions enable us to replace the binomials $a^2 - u^2$, $a^2 + u^2$, and $u^2 - a^2$ be single square terms. We can use:-

$$u = a\sin\theta \quad for \quad a^2 - u^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

$$u = a\tan\theta \quad for \quad a^2 + u^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

$$u = a\sec\theta \quad for \quad u^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$

EX-5 **Evaluate the following integrals:**

$$1)\ \int \frac{z^5\, dz}{\sqrt{1+z^2}} \qquad 4)\ \int \frac{x^2}{\sqrt{9-x^2}}dx$$

$$2)\ \int \frac{dx}{\sqrt{4+x^2}} \qquad 5)\ \int \frac{dt}{\sqrt{25t^2-9}}$$

$$3)\ \int \frac{dx}{4-x^2} \qquad 6)\ \int \frac{dy}{\sqrt{25+9y^2}}$$

Sol.-

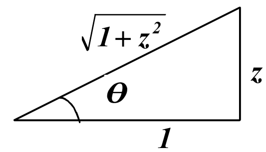
1) let $z = \tan\theta \quad \Rightarrow \quad dz = \sec^2\theta \cdot d\theta \qquad \tan\theta = \dfrac{z}{1}$

$$\int \frac{z^5\, dz}{\sqrt{1+z^2}} = \int \frac{\tan^5\theta \cdot \sec^2\theta\ d\theta}{\sqrt{1+\tan^2\theta}} = \int \tan^5\theta \cdot \sec\theta\ d\theta$$

$$= \int \tan\theta \cdot \sec\theta \left(\sec^2\theta - 1\right)^2 d\theta$$

$$= \int \sec^4\theta\left(\tan\theta \cdot \sec\theta\, d\theta\right) - 2\int \sec^2\theta\left(\tan\theta \cdot \sec\theta\, d\theta\right) + \int \tan\theta \cdot \sec\theta\, d\theta$$

$$= \frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta + c$$

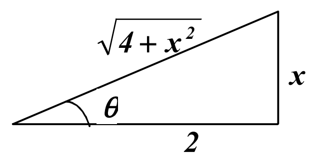


$$= \frac{1}{5}(\sqrt{1+z^2}\,)^5 - \frac{2}{3}(\sqrt{1+z^2}\,)^3 + \sqrt{1+z^2} + c$$

2) let $x = 2\tan\theta \quad \Rightarrow \quad dx = 2\sec^2\theta \cdot d\theta \qquad \tan\theta = \dfrac{x}{2}$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\ d\theta}{\sqrt{4+4\tan^2\theta}} = \int \sec\theta\ d\theta = \ln\left|\sec\theta + \tan\theta\right| + c$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

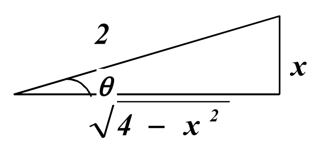


$$= \ln\left|\sqrt{4+x^2} + x\right| + c' \qquad \text{where } c' = c - \ln 2$$

3) let $x = 2\sin\theta \quad \Rightarrow \quad dx = 2\cos\theta \cdot d\theta$

$$\int \frac{dx}{4-x^2} = \int \frac{2\cos\theta\ d\theta}{4-4\sin^2\theta} = \frac{1}{2}\int \frac{d\theta}{\cos\theta} = \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + c$$



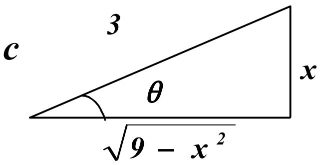
$$= \frac{1}{2}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{2+x}{\sqrt{(2-x)(2+x)}}\right| + c = \frac{1}{2}\ln\left|\sqrt{\frac{2+x}{2-x}}\right| + c = \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + c$$

4) let $x = 3\sin\theta \quad \Rightarrow \quad dx = 3\cos\theta \cdot d\theta$

$$\int \frac{x^2\,dx}{\sqrt{9 - x^2}} = \int \frac{9\sin^2\theta}{\sqrt{9 - 9\sin^2\theta}}\, 3\cos\theta\, d\theta = 9\int \sin^2\theta\, d\theta$$

$$= 9\int \frac{1 - \cos 2\theta}{2}\, d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + c$$



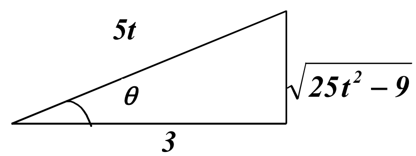
$$= \frac{9}{2}\left(\theta - \sin\theta \cdot \cos\theta\right) + c$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9 - x^2}}{3}\right) + c = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\cdot\sqrt{9 - x^2} + c$$

5) let $5t = 3\sec\theta \quad \Rightarrow \quad 5dt = 3\sec\theta \cdot \tan\theta\, d\theta$

$$\int \frac{dt}{\sqrt{25t^2 - 9}} = \int \frac{\frac{3}{5}\sec\theta \cdot \tan\theta\, d\theta}{\sqrt{9\sec^2\theta - 9}} = \frac{1}{5}\int \sec\theta\, d\theta$$

$$= \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + c$$



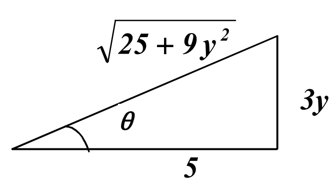
$$= \frac{1}{5}\ln\left|\frac{5t}{3} + \frac{\sqrt{25t^2 - 9}}{3}\right| + c$$

$$= \frac{1}{5}\ln\left|5t + \sqrt{25t^2 - 9}\right| + c' \quad \text{where } c' = c - \frac{1}{5}\ln 3$$

6) let $3y = 5\tan\theta \quad \Rightarrow \quad 3dy = 5\sec^2\theta\, d\theta$

$$\int \frac{dy}{\sqrt{25 + 9y^2}} = \int \frac{\frac{5}{3}\sec^2\theta\, d\theta}{\sqrt{25 + 25\tan^2\theta}} = \frac{1}{3}\int \sec\theta\, d\theta$$

$$= \frac{1}{3}\ln\left|\sec\theta + \tan\theta\right| + c$$



$$= \frac{1}{3}\ln\left|\frac{\sqrt{25 + 9y^2}}{5} + \frac{3y}{5}\right| + c$$

$$= \frac{1}{3}\ln\left|\sqrt{25 + 9y^2} + 3y\right| + c' \quad \text{where } c' = c - \frac{1}{3}\ln 5$$

EX-6 **Prove the following formulas:**

$$1)\ \int \frac{du}{\sqrt{a^2-u^2}} = sin^{-1}\frac{u}{a}+c \qquad 2)\ \int \frac{du}{a^2+u^2} = \frac{1}{a}tan^{-1}\frac{u}{a}+c$$

Proof.-

$$1)\quad let \quad u = a\,sin\theta \quad \Rightarrow \quad du = a\,cos\theta\cdot d\theta$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{a\,cos\theta\cdot d\theta}{\sqrt{a^2-a^2\,sin^2\theta}} = \int d\theta = \theta + c = sin^{-1}\frac{u}{a}+c$$

$$2)\quad let \quad u = a\,tan\theta \quad \Rightarrow \quad du = a\,sec^2\theta\cdot d\theta$$

$$\int \frac{du}{a^2+u^2} = \int \frac{a\,sec^2\theta\cdot d\theta}{a^2+a^2\,tan^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}tan^{-1}\frac{u}{a}+c$$

6-4- Integral involving ax^2+bx+c :

By using the algebraic process called completing the square, we can convert any quadratic: ax^2+bx+c , $a \neq 0$ to the form: $a(u^2 \mp A^2)$ we can then use one of the trigonometric substitutions to write the expression as a times a single square term.

EX-7 **– Evaluate:**

$$1)\ \int \frac{dx}{\sqrt{2x-x^2}} \qquad 4)\ \int \frac{dx}{\sqrt{1+x-x^2}}$$

$$2)\ \int \frac{dx}{2x^2+2x+1} \qquad 5)\ \int \frac{dx}{\sqrt{x^2-2x-8}}$$

$$3)\ \int \frac{dx}{\sqrt{x^2+2x+2}}$$

Sol.

$$1)\ \int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x^2-2x+1)}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$$

$$let \quad x-1 = sin\theta \quad \Rightarrow \quad dx = cos\theta\, d\theta$$

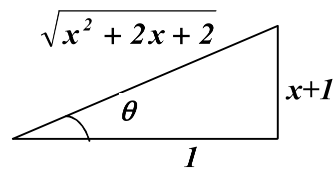
$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{cos\theta\, d\theta}{\sqrt{1-sin^2\theta}} = \int d\theta = \theta + c = sin^{-1}(x-1)+c$$

$$2)\ \int \frac{dx}{2x^2+2x+1} = \frac{1}{2}\int \frac{dx}{x^2+x+\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$let \quad x+\frac{1}{2} = \frac{1}{2}\tan\theta \quad \Rightarrow \quad dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$\int \frac{dx}{2x^2+2x+1} = \frac{1}{2}\int \frac{\frac{1}{2}\sec^2\theta\, d\theta}{\frac{1}{4}\tan^2\theta+\frac{1}{4}} = \int d\theta = \theta + c = \tan^{-1}(2x+1)+c$$

$$3)\ \int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{dx}{\sqrt{(x+1)^2+1}}$$



$$let \quad x+1 = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta\, d\theta$$

$$\int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{\tan^2\theta+1}} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta+\tan\theta\right| + c = \ln\left|\sqrt{x^2+2x+2}+x+1\right| + c$$

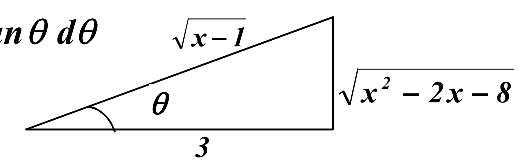
$$4)\ \int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{dx}{\sqrt{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}}$$

$$let \quad x-\frac{1}{2} = \frac{\sqrt{5}}{2}\sin\theta \quad \Rightarrow \quad dx = \frac{\sqrt{5}}{2}\cos\theta\, d\theta$$

$$= \int \frac{\frac{\sqrt{5}}{2}\cos\theta\, d\theta}{\sqrt{\frac{5}{4}-\frac{5}{4}\sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$$

$$5)\ \int \frac{dx}{\sqrt{x^2-2x-8}} = \int \frac{dx}{\sqrt{(x-1)^2-9}}$$

$$let \quad x-1 = 3\sec\theta \quad \Rightarrow \quad dx = 3\sec\theta\cdot\tan\theta\, d\theta$$



$$= \int \frac{3\sec\theta\cdot\tan\theta\, d\theta}{\sqrt{9\sec^2\theta-9}} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta+\tan\theta\right| + c = \ln\left|\frac{x-1}{3}+\frac{\sqrt{x^2-2x-8}}{3}\right| + c$$

$$= \ln\left|x-1+\sqrt{x^2-2x-8}\right| + c' \qquad where \quad c' = c - \ln 3$$