The solution to the differential equation is therefore

$$y = B \sin\left(\frac{n\pi x}{L}\right)$$

and the coefficient B is indeterminate. This result is a consequence of approximations made in formulating the differential equation; a linear representation of a nonlinear phenomenon was used.

For the usual case of a compression member with no supports between its ends, $n = 1$ and the Euler equation is written as

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{4.3}$$

It is convenient to rewrite Equation 4.3 as

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 EAr^2}{L^2} = \frac{\pi^2 EA}{(L/r)^2}$$

where A is the cross-sectional area and r is the radius of gyration with respect to the axis of buckling. The ratio L/r is the slenderness ratio and is the measure of a member's slenderness, with large values corresponding to slender members.

If the critical load is divided by the cross-sectional area, the critical buckling stress is obtained:

$$F_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L/r)^2} \tag{4.4}$$

At this compressive stress, buckling will occur about the axis corresponding to r. Buckling will take place as soon as the load reaches the value given by Equation 4.3, and the column will become unstable about the principal axis corresponding to the largest slenderness ratio. This axis usually is the axis with the smaller moment of inertia (we examine exceptions to this condition later). Thus the minimum moment of inertia and radius of gyration of the cross section should ordinarily be used in Equations 4.3 and 4.4.

EXAMPLE 4.1

A W12 × 50 is used as a column to support an axial compressive load of 145 kips. The length is 20 feet, and the ends are pinned. Without regard to load or resistance factors, investigate this member for stability. (The grade of steel need not be known: The critical buckling load is a function of the modulus of elasticity, not the yield stress or ultimate tensile strength.)

SOLUTION For a W12 × 50,

$$\text{Minimum } r = r_y = 1.96 \text{ in.}$$

$$\text{Maximum } \frac{L}{r} = \frac{20(12)}{1.96} = 122.4$$

$$P_{cr} = \frac{\pi^2 EA}{(L/r)^2} = \frac{\pi^2(29{,}000)(14.6)}{(122.4)^2} = 278.9 \text{ kips}$$

ANSWER Because the applied load of 145 kips is less than P_{cr}, the column remains stable and has an overall factor of safety against buckling of 278.9/145 = 1.92.

Early researchers soon found that Euler's equation did not give reliable results for stocky, or less slender, compression members. The reason is that the small slenderness ratio for members of this type causes a large buckling stress (from Equation 4.4). If the stress at which buckling occurs is greater than the proportional limit of the material, the relation between stress and strain is not linear, and the modulus of elasticity E can no longer be used. (In Example 4.1, the stress at buckling is $P_{cr}/A = 278.9/14.6 = 19.10$ ksi, which is well below the proportional limit for any grade of structural steel.) This difficulty was initially resolved by Friedrich Engesser, who proposed in 1889 the use of a variable tangent modulus, E_t, in Equation 4.3. For a material with a stress–strain curve like the one shown in Figure 4.5, E is not a constant for stresses greater than the proportional limit F_{pl}. The tangent modulus E_t is defined as the slope of the tangent to the stress–strain curve for values of f between F_{pl} and F_y. If the compressive stress at buckling, P_{cr}/A, is in this region, it can be shown that

$$P_{cr} = \frac{\pi^2 E_t I}{L^2} \tag{4.5}$$

Equation 4.5 is identical to the Euler equation, except that E_t is substituted for E.

FIGURE 4.5