

2.7 Integration methods/ powers of trigonometric function $\sin \theta$ & $\cos \theta$

Some formulas of integration

$$1. \int x^n dx = \frac{x^{n+1}}{n+1} + c$$

$$2. \int u(x)^n dx = \frac{u(x)^{n+1}}{n+1} + c$$

$$3. \int \frac{du}{u} dx = \ln |u| + c$$

$$4. \int e^u du = e^u + c$$

$$5. \int a^u du = \frac{1}{\ln a} a^u + c$$

$$6. \int \frac{du}{\sqrt{1-u^2}} = \left\{ \begin{array}{l} \sin^{-1}(u)+c \\ -\cos^{-1}(u)+c \end{array} \right\}$$

$$7. \int \frac{du}{1+u^2} = \left\{ \begin{array}{l} \tan^{-1}(u)+c \\ -\cot^{-1}(u)+c \end{array} \right\}$$

$$8. \int \frac{du}{u\sqrt{u^2-1}} = \left\{ \begin{array}{l} \sec^{-1}(u)+c \\ -\csc^{-1}(u)+c \end{array} \right\}$$

powers of trigonometric function $\sin \theta$ & $\cos \theta$.

First /// If odd powers of ($\sin \theta$ & $\cos \theta$)

→ using relation ($\sin^2 x = 1 - \cos^2 x$) or ($\cos^2 x = 1 - \sin^2 x$)

Example 1: Find $\int \cos^5 5x \, dx$.

Solution //

$$\begin{aligned}
 &= \int (\cos^2 5x)^2 \cos 5x \, dx = \int (1 - \sin^2 5x)^2 \cos 5x \, dx \\
 &= \int (1 - 2\sin^2 5x + \sin^4 5x) \cos 5x \, dx \\
 &= \int (\cos 5x \, dx - 2 \int \sin^2 5x \cdot \cos 5x \, dx + \int \sin^4 5x \cos 5x \, dx) \\
 &= \frac{\sin 5x}{5} - \frac{2}{5 \cdot 3} \sin^3 5x + \frac{\sin^5 5x}{25} + c
 \end{aligned}$$

Example 2: Find $\int \sin^5 x \cdot \cos^2 x \, dx$.

Solution //

$$\begin{aligned}
 &= \int (\sin^2 x)^2 \sin x \cos^2 x \, dx \\
 &= \int (1 - \cos^2 x)^2 \sin x \cos^2 x \, dx \\
 &= \int (1 - 2\cos^2 x + \cos^4 x) \sin x \cos^2 x \, dx \\
 &= \int (\sin x \cos^2 x \, dx - 2 \int \cos^4 x \sin x \, dx + \int \cos^6 x \sin x \, dx) \\
 &= \frac{-\cos^3 x}{3} + \frac{2}{5} \cos^5 x - \frac{\cos^7 x}{7} + c
 \end{aligned}$$

Exercises: 1. $\int \sin^3 x \, dx$ 2. $\int \sin^3 x \cos^2 x \, dx$

Second /// If even powers of ($\sin \theta$ & $\cos \theta$)

→ using relation ($\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ or ($\cos^2 x = \frac{1}{2}(1 + \cos 2x)$)

Example: Find $\int \cos^2 2x \, dx$.

Solution //

$$= \int \frac{1}{2}(1 + \cos 4x) \, dx = \frac{x}{2} + \frac{\sin 4x}{8} + c$$

Example : Find $\int \sin^4 3x \, dx$

Solution //

$$\begin{aligned} \int (\sin^2 3x)^2 \, dx &= \int \left[\frac{1}{2}(1 - \cos 6x) \right]^2 \, dx \\ &= \frac{1}{4} \int (1 - 2\cos 6x + \cos^2 6x) \, dx \\ &= \frac{1}{4} \left(x - \frac{1}{3} \sin 6x + \int \frac{1}{2}(1 + \cos 12x) \, dx \right) \\ &= \frac{1}{4} x - \frac{1}{12} \sin 6x + \frac{1}{8} x + \frac{1}{4} \frac{1}{2} \frac{1}{12} \sin 12x + c \\ &= \frac{1}{4} x - \frac{1}{12} \sin 6x + \frac{1}{8} x + \frac{1}{96} \sin 12x + c \end{aligned}$$

Example : Find $\int \sin^2 x \cos^2 x \, dx$

Solution //

$$\begin{aligned} &= \int \frac{1}{2}(1 - \cos 2x) \frac{1}{2}(1 + \cos 2x) \, dx \\ &= \int \frac{1}{4} [(1 + \cos 2x - \cos 2x - \cos^2 2x)] \, dx \end{aligned}$$

$$\begin{aligned}\int \frac{1}{4} [(1 - \cos^2 2x)] dx &= \int \frac{1}{4} \left[\left(1 - \frac{1}{2}(1 + \cos 4x)\right) \right] dx \\ &= \frac{1}{4} \left[\left(x - \frac{x}{2} - \frac{1}{8} \sin 4x\right) \right] + c = \frac{x}{8} - \frac{1}{32} \sin 4x + c\end{aligned}$$

Exercise: $\int \sin^4 x \, dx$

Third /// If multiplication of $\sin \theta$ & $\cos \theta$ with different angles

→ using relations

$$1 - \sin mx \sin nx = \frac{1}{2} [\cos(m-n)x - \cos(m+n)x]$$

$$2 - \sin mx \cos nx = \frac{1}{2} [\sin(m-n)x + \sin(m+n)x]$$

$$3 - \cos mx \cos nx = \frac{1}{2} [\cos(m-n)x + \cos(m+n)x]$$

Example : Find $\int \sin 3x \cos 5x \, dx$

Solution /

$$\begin{aligned}&= \frac{1}{2} \int [\sin(3-5)x + \sin(3+5)x] dx \\ &= \frac{1}{2} \int [\sin(-2)x + \sin(8)x] dx = \frac{1}{4} \cos 2x - \frac{1}{16} \cos 8x + c\end{aligned}$$

Exercise: $\int_0^\pi \cos 3x \cdot \cos 4x \, dx$