كلية العلوم

قسم الأمن السيبراني

Lecture: (4)

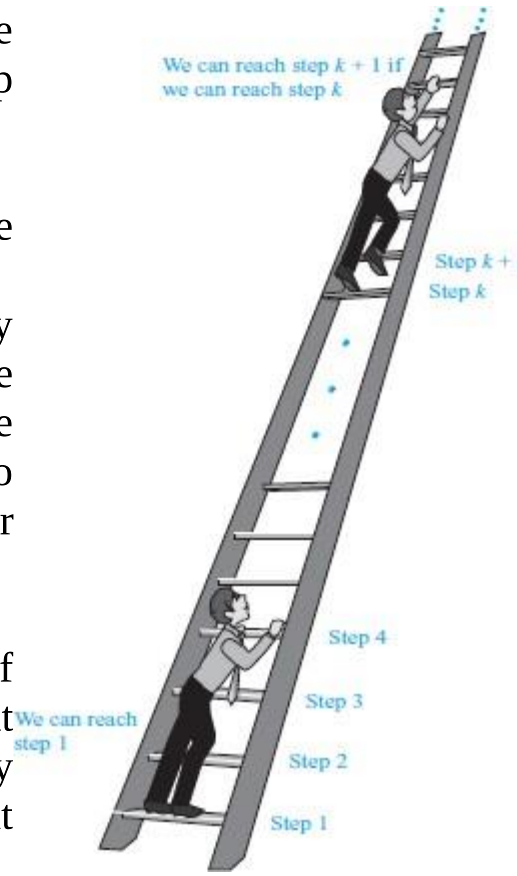
Mathematic induction

Subject: Discrete Structures

First Stage: Semester II

Lecturer: BAQER KAREEM SALIM

Mathematic induction:



Suppose that we have an infinite ladder and we want to know whether we can reach every step on this ladder. We know two things:
l. We can reach the first rung of the ladder.
2. If we can reach a particular rung of the ladder, then we can reach the next rung.
Can we conclude that we can reach every rung? By (1), we know that we can reach the first rung of the ladder. Moreover, because we can reach the first rung, by (2), we can also reach the second rung; it is the next rung after the first rung. Applying

We can verify using an important proof technique called mathematical induction. That is, we can show that *P(n)* is true for every positive integer *n*, where *P(n)* is the statement that we can reach the *nth* rung of the ladder.
Mathematical induction is an important proof technique that can be used to prove assertions of this type. Mathematical induction is used to prove results about a large variety of discrete objects. For example, it is used to prove results about the complexity of algorithms, the correctness of certain types of computer programs, theorems about graphs and trees, as well as a wide range of identities and inequalities.
In general, mathematical induction can be used to prove statements that assert that *P(n)* is true for all positive integers *n,* where *P(n)* is a propositional function.

PRINCIPLE OF *MATHEMATICAL INDUCTION*

To prove that *P(n)* is true for all positive integers *n*, where *P(n)* is a propositional function, we complete two steps:

(i) BASIS *STEP:* We verify that *P(1)* is true.

(ii) INDUCTIVE STEP: We show that the conditional statement $P(k) \rightarrow P(k+1)$ is true for all positive integers *k*.

EXAMPLE1:

Show that if *n* is a positive integer, then

$$1+2+\ldots+n = \frac{n(n+1)}{2}$$

Prove P (for $n \geq 1$)

Solution:

Let *P(n)* be the proposition that the sum of the first *n* positive integers is $n(n+1)/2$

We must do two things to prove that *P(n)* is true for $n = 1, 2, 3, \ldots$.

Namely, we must show that *P (1)* is true and that the conditional statement *P(k)* implies $P(k+1)$ is true for $k = 1,2.3, \ldots$.

(i) BASIS STEP: P (1) IS true, because $1 = \frac{1(1+1)}{2}$

left side =1 & Right side =2/2 = 1

left side = Right side

(ii) *INDUCTIVE STEP:* For the inductive hypothesis we assume that *P(k)* holds for an arbitrary positive integer *k*. That is, we assume that P(k) is true

$$1+2+\dots+k = \frac{k(k+1)}{2}$$

Under this assumption, it must be shown that *P(k* + 1) is true, namely, that

to prove that P(k+1) is true

$$1 + 2 + 3 + 4 + \dots + k + (k+1) = 1/2 * k * (k+1) + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= 1/2\,(k+1)(k+2)$$

So, P is true for all $n \geq k$

Example 2:

Conjecture a formula for the sum of the first *n* positive odd integers. Then prove your conjecture using mathematical induction.

Solution:

The sums of the first *n* positive odd integers for $n = 1,2,3,4,5$ are:

$1 = 1$, $\quad 1 + 3 = 4$, $\quad 1 + 3 + 5 = 9$,
$1+3+5+7=16$, $\quad 1 + 3 + 5 + 7 + 9 = 25$.

From these values it is reasonable to conjecture that the sum of the first *n* positive odd integers is n^2, that is,

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

We need a method to *prove* that this *conjecture* is correct, if in fact it is.

Let $P(n)$ denote the proposition that the sum of the first *n* odd positive integers is n^2

(i) *BASIS STEP: P(1)* states that the sum of the first one odd positive integer is 1^2. This is true because the sum of the first odd positive integer is 1.

(ii) *INDUCTIVE STEP:*

we first assume the inductive hypothesis.

The inductive hypothesis is the statement that $P(k)$ is true, that is,

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2$$

(ii) n=k; Assuming P(k) is true,

We add (2k-1)+2 = 2K + 1 to both sides of P(k), obtaining:

$$1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$
$$= (k + 1)^2$$

Which is P(k + 1).
That is, P(k + 1) is true whenever P(k) is true. By the principle of mathematical induction:

P is true for all $n \geq .k$.

Example 3:
Prove the following proposition (for $n \geq 0$):

$$P(n): 1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$$

solution:

(i) P(0) : left side =1

Right side $=2^1-1=1$

(ii) Assuming P(k) is true; n=k

$$P(k) : 1 + 2 + 2^2 + 2^3 + \ldots + 2^k = 2^{k+1} - 1$$

We add 2^{k+1} to both sides of P(k), obtaining

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$
$$= 2(2^{k+1}) - 1 = 2^{k+2} - 1$$

which is P(k +1). That is, P(k +1) is true whenever P(k) is true.
By the principle of induction:

P(n) is true for all n.

Homework:
Prove by induction:

1) $2 + 4 + 6 + \ldots\ldots + 2n = n(n + 1)$
2) $1 + 4 + 7 + \ldots\ldots + (3n - 2) = 1/2\, n(3n - 1)$