

2.8 Integration methods/trigonometric substitution

Formulas

$$\left(\begin{array}{c} a^2 + u^2 \\ \text{or} \\ \sqrt{a^2 + u^2} \end{array} \right)$$

$\tan \theta$

$$\left(\begin{array}{c} a^2 - u^2 \\ \text{or} \\ \sqrt{a^2 - u^2} \end{array} \right)$$

$\sin \theta$

$$\left(\begin{array}{c} u^2 - a^2 \\ \text{or} \\ \sqrt{u^2 - a^2} \end{array} \right)$$

$\sec \theta$

Steps of solution

1. Assume $u = a \sin \theta$ or $u = a \tan \theta$ or $u = a \sec \theta$
2. We derive the two parties
3. After that we replace the assumed and derived function in the original function

Example 1: Find $\int \frac{1}{3^2 + x^2} dx$.

Solution //

$$1. u = a \tan \theta \rightarrow x = 3 \tan \theta \rightarrow \theta = \tan^{-1} \frac{x}{3}$$

$$2. dx = 3 \sec^2 \theta d\theta$$

$$\begin{aligned}
 3. &= \int \frac{1}{3^2 + (3 \tan \theta)^2} 3 \sec^2 \theta d\theta = \\
 &\int \frac{1}{9(1 + \tan^2 \theta)} 3 \sec^2 \theta d\theta \\
 &= \int \frac{\sec^2 \theta}{3 \sec^2 \theta} d\theta = \frac{1}{3} \theta + c = \frac{1}{3} \tan^{-1} \frac{x}{3} + c
 \end{aligned}$$

Example 2: Find $\int \frac{x^2}{\sqrt{9-x^2}} dx$.

Solution //

$$1. u = a \sin \theta \rightarrow x = 3 \sin \theta \rightarrow \theta = \sin^{-1} \frac{x}{3}$$

$$2. dx = 3 \cos \theta d\theta$$

$$\begin{aligned}
 3. &= \int \frac{(3 \sin \theta)^2 3 \cos \theta}{\sqrt{9 - (3 \sin \theta)^2}} d\theta = \int \frac{(3 \sin \theta)^2 3 \cos \theta}{\sqrt{9(1 - \sin^2 \theta)}} d\theta = \\
 &\frac{27 \sin^2 \theta \cos \theta}{3 \sqrt{1 - \sin^2 \theta}} = 9 \int \sin^2 \theta d\theta = 9 \int \frac{1}{2} (1 - \cos 2\theta) d\theta = \\
 &\frac{9}{2} \theta - \frac{9}{4} \sin 2\theta + c
 \end{aligned}$$

$$= \frac{9}{2} \theta - \frac{9}{4} 2 \sin \theta \cos \theta + c$$

$$= \frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{9}{2} * \frac{x}{3} * \frac{\sqrt{9-x^2}}{3} + c$$

Example 3: Find $\int \frac{1}{\sqrt{e^{2x}-1}} dx$

Solution// $\int \frac{1}{\sqrt{(e^x)^2-1}} dx$

1. $u = a \sec \theta \rightarrow e^x = \sec \theta \rightarrow \theta = \sec^{-1} e^x$

2. $e^x dx = \sec \theta \tan \theta d\theta$

3. $= \int \frac{\sec \theta \tan \theta}{e^x \sqrt{\sec^2 \theta - 1}} d\theta =$

$$\int \frac{\sec \theta \cdot \tan \theta}{\sec \theta \sqrt{\sec^2 \theta - 1}} d\theta = \int \frac{\sec \theta \cdot \tan \theta}{\sec \theta \cdot \tan \theta} d\theta = \theta + c =$$

$\sec^{-1} e^x + c$

Exercises: 1. $\int e^x \sqrt{1 - e^{2x}} dx$

2. $\int \frac{1}{4+x^2} dx$

3. $\int \frac{1}{x\sqrt{x^2-1}} dx$