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Lecture: (6)

Properties of relations, Composition of relations

Subject: Discrete Structures

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Properties of binary relations (Types of relations)

Let R be a relation on the set A

1) Reflexive :

R is said to be *reflexive* if ordered couple $(x, x) \in R$ for $\forall x \in X$.

$\forall a \in A \rightarrow aRa$ or $(a,a) \in R$; $\forall a, b \in A$. .
Thus R is not reflexive if there exists $a \in A$ such that
$(a, a) \notin R$.

Example i:

Consider the following five relations on the set $A = \{1, 2, 3, 4\}$:

$R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$

$R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R3 = \{(1, 3), (2, 1)\}$

$R4 = \emptyset$, the empty relation
$R5 = A \times A$, the universal relation

Determine which of the relations are reflexive.

Since A contains the four elements 1, 2, 3, and 4,
a relation R on A is reflexive if it contains the four pairs
$(1, 1), (2, 2), (3, 3)$, and $(4, 4)$.

Thus only $R2$ and the universal relation $R5 = A \times A$ are reflexive.
Note that $R1$,$R3R3$, and $R4$ are not reflexive since, for example, $(2, 2)$ does not belong to any of them.

Example ii

Consider the following five relations:

(1) Relation $\leq$ (less than or equal) on the set **Z** of integers.

(2) Set inclusion $\subseteq$ on a collection *C* of sets.

(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.

(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.

Determine which of the relations are reflexive.

The relation (3) is not reflexive since no line is perpendicular to itself. Also (4) is not reflexive since no line is parallel to itself. The other relations are reflexive; that is,

$x \leq x$ for every $x \in Z$,

$A \subseteq A$ for any set $A \in C$, and

2) **Symmetric** :

R is said to be *symmetric* if, ordered couple $(x, y) \in R$ and also ordered couple $(y, x) \in R$ for $\forall x, \forall y \in X$.

aRb $\rightarrow$ bRa $\forall$ a,b $\in$A. [if whenever $(a, b) \in R$ then $(b, a) \in R$.]

Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

Example

(a) Determine which of the relations in Example i are symmetric

$R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$

$R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R3 = \{(1, 3), (2, 1)\}$

$R4 = \emptyset$, *the empty relation*

$R5 = A \times A$, *the universal relation*

R1 is not symmetric since $(1, 2) \in R1$ but $(2, 1) \notin R1$.
R3 is not symmetric since $(1, 3) \in R3$ but $(3, 1) \notin R3$.
The other relations are symmetric.

(b) Determine which of the relations in Example ii are symmetric.

(1) Relation ≤ (less than or equal) on the set **Z** of integers.
(2) Set inclusion ⊆ on a collection *C* of sets.
(3) Relation ⊥ (perpendicular) on the set *L* of lines in the plane.
(4) Relation ∥ (parallel) on the set *L* of lines in the plane.

The relation ⊥ is symmetric since if line *a* is perpendicular to line *b* then *b* is perpendicular to *a*.
Also, ∥ is symmetric since if line *a* is parallel to line *b* then *b* is parallel to line *a*.
The other relations are not symmetric. For example:
$3 \leq 4$ but 4 not≤ 3; $\{1, 2\} \subseteq \{1, 2, 3\}$ but $\{1, 2, 3\}$ not $\subseteq \{1, 2\}$.

3) **Transitive** :
R is said to be *transitive* if ordered couple $(x, z) \in R$ whenever both ordered couples $(x, y) \in R$ and $(y, z) \in R$.

$aRb \wedge bRc \rightarrow aRc$. that is, if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$.

Thus R is not transitive if there exist $a, b, c \in R$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

Example

(a) Determine which of the relations in example i are transitive.

$R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$
$R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
$R3 = \{(1, 3), (2, 1)\}$
$R4 = \emptyset$, *the empty relation*
$R5 = A \times A$, *the universal relation*

The relation R3 is not transitive since $(2, 1), (1, 3) \in R3$ but $(2, 3) \notin R3$. All the other relations are transitive.

(b) Determine which of the relations in example ii are transitive.

(1) Relation $\leq$ (less than or equal) on the set Z of integers.

(2) Set inclusion $\subseteq$ on a collection C of sets.

(3) Relation $\perp$ (perpendicular) on the set L of lines in the plane.

(4) Relation $\parallel$ (parallel) on the set L of lines in the plane.

The relations $\leq$, $\subseteq$, and $|$ are transitive, but certainly not $\perp$.
Also, since no line is parallel to itself, we can have $a \parallel b$ and $b \parallel a$, but $a \parallel a$. Thus $\parallel$ is not transitive.

4) Equivalence relation:

A binary relation on any set is said an equivalence relation if it is **reflexive**, **symmetric**, and **transitive**.

R is an equivalence relation on *S* if it has the following three properties:

a - For every $a \in S$, aRa. (reflexive)
b- If aRb, then bRa. (symmetric)
c- If aRb and bRc, then aRc. (transitive)

5) Irreflexive:

$\forall\ a \in A\ (a,a) \notin R$

6) AntiSymmetric :

if $(x, y) \in R$ but $(y,x) \notin R$ unless $x = y$.
or
if aRb and bRa then a=b,

if $a \neq b$ and aRb then $(b,a) \notin R$. that is,

Thus, *R* is not antisymmetric if there exist distinct elements *a* and *b* in *A* such that *aRb* and *bRa*.

the relations $\geq, \leq$ and $\subseteq$ are antisymmetric

Example

(a) Determine which of the relations in Example i are antisymmetric.

$R1 = \{(1, 1), (1, 2), (2, 3), (1, 3), (4, 4)\}$
$R2 = \{(1, 1)(1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$
$R3 = \{(1, 3), (2, 1)\}$

$R4 = \emptyset$, the empty relation
$R5 = A \times A$, the universal relation

*R*2 is not antisymmetric since $(1, 2)$ and $(2, 1)$ belong to *R*2, but $1 \neq 2$. Similarly,
the universal relation *R*3 is not antisymmetric.
All the other relations are antisymmetric.

(b) Determine which of the relations in Example ii are antisymmetric.
(1) Relation $\leq$ (less than or equal) on the set **Z** of integers.
(2) Set inclusion $\subseteq$ on a collection *C* of sets.
(3) Relation $\perp$ (perpendicular) on the set *L* of lines in the plane.
(4) Relation $\parallel$ (parallel) on the set *L* of lines in the plane.

The relation $\leq$ is antisymmetric since whenever $a \leq b$ and $b \leq a$ then $a = b$.
Set inclusion $\subseteq$ is antisymmetric since whenever
$A \subseteq B$ and $B \subseteq A$ then $A = B$. Also,
The relations $\perp$ and $\parallel$ are not antisymmetric.

7) Compatible:
if a relation is only **reflexive** and **symmetric** then it is called a *compatibility* relation. So, we can say that: every equivalence relation is a compatibility relation, but not every compatibility relation is an equivalence relation.

Example:
Determine the properties of the relation $\subset$ of set (inclusion on any collection of sets):
1) $A \subset A$ for any set, so $\subset$ is reflexive

2) $A \subset B$ does not imply $B \subset A$, so $\subset$ is not symmetric
3) If $A \subset B$ and $B \subset C$ then $A \subset C$, so $\subset$ is transitive
4) $\subset$ is reflexive, not symmetric & transitive, so $\subset$ is not equivalence relations
5) $A \subset A$, so $\subset$ is not Irreflexive
6) If $A \subset B$ and $B \subset A$ then $A = B$, so $\subset$ is anti-symmetric
7) $\subset$ *is reflexive and not symmetric then it is not compatibility* relation.

Example:
If A ={1,2,3} and R={(1,1),(1,2),(2,1),(2,3)}, is R equivalence relation ?
1) 2 is in A but $(2,2) \notin R$, so R is not reflexive
2) $(2,3) \in R$ but $(3,2) \notin R$, so R is not symmetric
3) $(1,2) \in R$ and $(2,3) \in R$ but $(1,3) \notin R$, so R is not transitive
So R is not Equivalence relation.

Example:
What is the properties of the relation = ?
1) a=a for any element $a \in A$, so = is reflexive
2) If a = b then b = a, so = is symmetric
3) If a = b and b = c then a = c, so = is transitive
4) = is (reflexive + symmetric + transitive), so = is equivalence
5) a = a, so = is not Irreflexive
6) If a = b and b = a then a = b, so = is anti-symmetric
7) = *is reflexive and symmetric then it is compatibility* relation.

Remark:
The properties of being symmetric and being antisymmetric are not negatives of each other.
For example,
the relation R = {(1, 3), (3, 1), (2, 3)} is neither symmetric nor antisymmetric.
On the other hand, the relation R = {(1, 1), (2, 2)} is both symmetric and antisymmetric.

From the directed graph of a relation, we can easily examine some of its properties. For example, if a relation is reflexive, then we must get a self-loop at each node. Conversely if a relation is irreflexive, then there is no self-loop at any node.
For symmetric relation if one node is connected to another, then there must be a return arc from second node to the first node.
For antisymmetric relation there is no such direct return arc exist.
Similarly, we examine the transitivity of the relation in the directed graph.

8) Partial ordered relation
A binary relation R is said to be partial ordered relation if it is:
reflexive, antisymmetric, and transitive.

Example,
R={(w,w), (x, x), (y, y), (z, z),(w, x), (w, y), (w, z),(x, y), (x, z)}
In a partial ordered relation objects are related through superior/inferior criterion..

Example
In the arithmetic relation less than or equal to" ≤" (or greater than or equal to "≥") are partial ordered relations.

Since,

(1) Every number is equated to itself so it is reflexive.

(2) Also, if m and n are two numbers then ordered couple $(m, n) \in R$ if $m = n \Rightarrow n \nleq m$ so $(n, m) \notin R$ hence, relation is antisymmetric.

(3) if $(m, n) \in R$ and $(n, k) \in R \Rightarrow m = n$ and $n = k \Rightarrow m = k$ so $(m, k) \in R$ hence, R is transitive.

Example

The relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets since set inclusion has the three desired properties. That is,

(1) $A \subseteq A$ for any set A (reflexive).

(2) If $A \subseteq B$ and $B \subseteq A$, then $A = B$ (antisymmetric).

(3) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ (transitive).