Chapter six

Methods of integration

6-1- <u>Integration by parts</u>:

The formula for integration by parts comes from the product rule:-

$$d(u\cdot v)=u\cdot dv+v\cdot du \quad\Rightarrow\quad u\cdot dv=d(u\cdot v)-v\cdot du$$

and integrated to give: $\int u\,dv=\int d(u\cdot v)-\int v\,du$

then the integration by parts formula is:-

$$\int u\,dv=u\cdot v-\int v\,du$$

Rule for choosing u **and** dv **is:**

For u**: choose something that becomes simpler when differentiated.**

For dv**: choose something whose integral is simple.**

It is not always possible to follow this rule, but when we can.

<u>EX-1</u> **– Evaluate the following integrals:**

1) $\int xe^{x}\,dx$

2) $\int x\cdot \cos x\,dx$

3) $\int \frac{x}{\sqrt{x-1}}\,dx$

4) $\int x^{2}\cdot \ln x\,dx$

5) $\int x\cdot \sec^{2}x\,dx$

6) $\int \ln\left(x+\sqrt{1+x^{2}}\right)dx$

7) $\int \sin^{-1}ax\,dx$

8) $\int e^{ax}\cdot \sin bx\,dx$

9) $\int x^{3}\cdot e^{x}\,dx$

10) $\int x^{3}\cdot e^{x^{2}}\,dx$

<u>Sol.</u> **–**

1) let $\left.\begin{array}{l} u=x \Rightarrow du=dx \\ dv=e^{x}dx \Rightarrow v=e^{x}\end{array}\right\}\Rightarrow \int u\,dv=u\cdot v-\int v\,du$

$$\int x\cdot e^{x}dx=x\cdot e^{x}-\int e^{x}dx=x\cdot e^{x}-e^{x}+c$$

$$2)\quad let\quad \left.\begin{array}{l} u = x \quad \Rightarrow \quad du = dx \\ dv = \cos x\, dx \;\Rightarrow\; v = \sin x \end{array}\right\} \Rightarrow \quad \int u\,dv = u \cdot v - \int v\,du$$

$$\int x \cdot \cos x\, dx = x \cdot \sin x - \int \sin x\, dx = x \cdot \sin x + \cos x + c$$

$$3)\quad let\quad \left.\begin{array}{l} u = x \quad \Rightarrow \quad du = dx \\ dv = \dfrac{1}{\sqrt{x-1}}dx \;\Rightarrow\; v = 2(x-1)^{1/2} \end{array}\right\} \Rightarrow \quad \int u\,dv = u \cdot v - \int v\,du$$

$$\int \frac{x}{\sqrt{x-1}}dx = 2x \cdot (x-1)^{1/2} - 2\int (x-1)^{1/2}dx$$

$$= 2x \cdot \sqrt{x-1} - \frac{2(x-1)^{3/2}}{3/2} + c = 2x \cdot \sqrt{x-1} - \frac{4}{3}\sqrt{(x-1)^3} + c$$

$$4)\quad let\quad \left.\begin{array}{l} u = \ln x \quad \Rightarrow \quad du = \dfrac{1}{x}dx \\ dv = x^2 dx \;\Rightarrow\; v = \dfrac{x^3}{3} \end{array}\right\} \Rightarrow \quad \int u\,dv = u \cdot v - \int v\,du$$

$$\int x^2 \cdot \ln x\, dx = \frac{x^3}{3} \cdot \ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3} \cdot \ln x - \frac{1}{9}x^3 + c$$

$$5)\quad let\quad \left.\begin{array}{l} u = x \quad \Rightarrow \quad du = dx \\ dv = \sec^2 x\, dx \;\Rightarrow\; v = \tan x \end{array}\right\} \Rightarrow \quad \int u\,dv = u \cdot v - \int v\,du$$

$$\int x \cdot \sec^2 x\, dx = x \cdot \tan x - \int \tan x\, dx = x \cdot \tan x + \ln|\cos x| + c$$

$$6)\quad let\quad u = \ln\left(x + \sqrt{1+x^2}\right) \;\Rightarrow\; du = \frac{1 + \dfrac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}}\,dx$$

$$dv = dx \quad \Rightarrow \quad v = x$$

$$\int \ln\left(x + \sqrt{1+x^2}\right)dx = x \cdot \ln\left(x + \sqrt{1+x^2}\right) - \int x\left(1+x^2\right)^{-1/2}dx$$

$$= x \cdot \ln\left(x + \sqrt{1+x^2}\right) - \frac{1}{2} \cdot \frac{\left(1+x^2\right)^{1/2}}{1/2} + c = x \cdot \ln\left(x + \sqrt{1+x^2}\right) - \sqrt{1+x^2} + c$$

7) let $u=\sin^{-1}ax \Rightarrow du=\dfrac{a\,dx}{\sqrt{1-a^2x^2}}$ & $dv=dx \Rightarrow v=x$

$$\int \sin^{-1}ax\,dx = x\cdot\sin^{-1}ax-\int\frac{a\,x}{\sqrt{1-a^2x^2}}dx$$

$$= x\cdot\sin^{-1}ax+\frac{1}{2a}\int -2a^2x\left(1-a^2x^2\right)^{-1/2}dx$$

$$= x\cdot\sin^{-1}ax+\frac{1}{2a}\cdot\frac{\left(1-a^2x^2\right)^{1/2}}{1/2}+c = x\cdot\sin^{-1}ax+\frac{\sqrt{1-a^2x^2}}{a}+c$$

8) let $u=e^{ax} \Rightarrow du=a\cdot e^{ax}\,dx$ & $dv=\sin bx\,dx \Rightarrow v=-\dfrac{1}{b}\cos bx$

$$\int e^{ax}\cdot\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b}\int e^{ax}\cdot\cos bx\,dx \quad ....(1)$$

let $u=e^{ax} \Rightarrow du=a\cdot e^{ax}\,dx$ & $dv=\cos bx\,dx \Rightarrow v=\dfrac{1}{b}\sin bx$

$$\int e^{ax}\cdot\cos bx\,dx=\frac{1}{b}e^{ax}\cdot\sin bx-\frac{a}{b}\int e^{ax}\cdot\sin bx\,dx \quad ....(2)$$

sub. (2) in (1) $\Rightarrow$

$$\int e^{ax}\cdot\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx\,dx-\frac{a^2}{b^2}\int e^{ax}\cdot\sin bx\,dx$$

$$\int e^{ax}\cdot\sin bx\,dx+\frac{a^2}{b^2}\int e^{ax}\cdot\sin bx\,dx=-\frac{1}{b}e^{ax}\cdot\cos bx+\frac{a}{b^2}e^{ax}\cdot\sin bx\,dx+c$$

$$\left(1+\frac{a^2}{b^2}\right)\int e^{ax}\cdot\sin bx\,dx=\frac{e^{ax}}{b^2}\left(a\sin bx-b\cos bx\right)+c$$

$$\therefore\quad \int e^{ax}\cdot\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)+c$$

9) <u>derivative of u</u> <u>integration of dv</u>

derivative of u	sign	integration of dv
x^3	+	e^x
$3x^2$	-	e^x
$6x$	+	e^x
6	-	e^x
0		e^x

$$\therefore \int x^3e^{ax}\,dx = x^3e^x-3x^2e^x+6xe^x-6e^x+c$$

$$=e^x\left(x^3-3x^2+6x-6\right)+c$$

$$10)\quad let \quad u = x^2 \quad \Rightarrow \quad du = 2x\,dx \quad \& \quad dv = x \cdot e^{x^2}dx \Rightarrow v = \frac{1}{2}e^{x^2}$$

$$\int x^3 \cdot e^{x^2}dx = \frac{1}{2}x^2 \cdot e^{x^2} - \frac{1}{2}\int 2x \cdot e^{x^2}dx = \frac{1}{2}x^2 \cdot e^{x^2} - \frac{1}{2}e^{x^2} + c$$

6-2- Odd and even powers of sine and cosine:

To integrate an odd positive power of $sinx$ (say $sin^{2n+1}x$) we split off a factor of $sinx$ and rewrite the remaining even power in terms of the cosine. We write:-

$$\int sin^{2n+1} x \cdot dx = \int \left(1 - cos^2 x\right)^n \cdot sin\,x\ dx$$

$$and \int cos^{2n+1} x \cdot dx = \int \left(1 - sin^2 x\right)^n \cdot cos\,x\ dx$$

***EX-2*- Evaluate:**

$1) \int sin^3 x\,dx$ $\qquad$ $2) \int cos^5 x\,dx$

***Sol.*-**

$$1) \int sin^3 x\,dx = \int sin^2 x \cdot sinx\,dx = \int \left(1 - cos^2 x\right) \cdot sinx\,dx$$

$$= \int sinx\,dx + \int cos^2 x \cdot \left(-sinx\right)dx = -cos\,x + \frac{1}{3}cos^3 x + c$$

$$2) \int cos^5 x\,dx = \int cos^4 x \cdot cosx\ dx = \int \left(1 - sin^2 x\right)^2 \cdot cos\,x\,dx$$

$$= \int cosx\,dx - 2\int sin^2 x \cdot cos\,x\,dx + \int sin^4 x \cdot cos\,x\,dx$$

$$= sinx - \frac{2}{3}sin^3 x + \frac{1}{5}sin^5 x + c$$

To integrate an even positive power of sine (say $sin^{2n}x$) we use the relations:-

$$cos^2\theta = \frac{1 + cos\,2\theta}{2} \qquad or \qquad sin^2\theta = \frac{1 - cos\,2\theta}{2}$$

then we can write:-

$$\int \sin^{2n} x \cdot dx = \int \left(\frac{1-\cos 2x}{2} \right)^n dx$$

$$\text{and } \int \cos^{2n} x \cdot dx = \int \left(\frac{1+\cos 2x}{2} \right)^n dx$$

EX-3- **Evaluate:**

$1)\ \int \cos^2 \theta \, d\theta$ $\qquad\qquad$ $2)\ \int \sin^4 \theta \, d\theta$

Sol.-

$$1)\ \int \cos^2\theta \, d\theta = \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}\left[\int d\theta + \frac{1}{2}\int 2\cos 2\theta \, d\theta \right]$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta \right] + c$$

$$2)\ \int \sin^4\theta \, d\theta = \int \left(\frac{1-\cos 2\theta}{2} \right)^2 d\theta = \frac{1}{4}\left[\int d\theta - \int \cos 2\theta (2d\theta) + \int \cos^2 2\theta \, d\theta \right]$$

$$= \frac{1}{4}\left[\theta - \sin 2\theta + \int \frac{1+\cos 4\theta}{2} d\theta \right] = \frac{1}{4}\left[\theta - \sin 2\theta + \frac{1}{2}(\theta + \frac{1}{4}\sin 4\theta) \right] + c$$

$$= \frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + c$$

To integrate the following identities:-

$$\int \sin mx \cdot \sin nx \ dx \ , \ \int \sin mx \cdot \cos nx \ dx \ , \text{and } \int \cos mx \cdot \cos nx \ dx$$

we use the following formulas:-

$$\sin mx \cdot \sin nx = \frac{\cos(m-n)x - \cos(m+n)x}{2}$$

$$\sin mx \cdot \cos nx = \frac{\sin(m-n)x + \sin(m+n)x}{2}$$

$$\cos mx \cdot \cos nx = \frac{\cos(m-n)x + \cos(m+n)x}{2}$$

EX-4- **Evaluate:**

$$1)\ \int \sin 3x \cdot \cos 5x\, dx \qquad 2)\ \int \cos x \cdot \cos 7x\, dx \qquad 3)\int \sin x \cdot \sin 2x\, dx$$

Sol.-

$$1)\ \int \sin 3x \cdot \cos 5x\, dx = \frac{1}{2}\int \left(\sin(3x-5x)+\sin(3x+5x)\right)dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}\int \sin 2x(2dx) + \frac{1}{8}\int \sin 8x(8dx)\right] = \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + c$$

$$2)\ \int \cos x \cdot \cos 7x\, dx = \frac{1}{2}\int \left(\cos(6x)+\cos(8x)\right)dx = \frac{1}{12}\sin 6x + \frac{1}{16}\sin 8x + c$$

$$3)\ \int \sin x \cdot \sin 2x\, dx = \frac{1}{2}\int \left(\cos x - \cos 3x\right)dx = \frac{1}{2}\sin x - \frac{1}{6}\sin 3x + c$$

6-3- Trigonometric substitutions:

Trigonometric substitutions enable us to replace the binomials a^2-u^2 , a^2+u^2 , and u^2-a^2 be single square terms. We can use:-

$$u = a\sin\theta \quad for \quad a^2-u^2 = a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$$

$$u = a\tan\theta \quad for \quad a^2+u^2 = a^2 + a^2\tan^2\theta = a^2(1+\tan^2\theta) = a^2\sec^2\theta$$

$$u = a\sec\theta \quad for \quad u^2-a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$$

EX-5 **Evaluate the following integrals:**

$$1)\ \int \frac{z^5\, dz}{\sqrt{1+z^2}} \qquad 4)\ \int \frac{x^2}{\sqrt{9-x^2}}dx$$

$$2)\ \int \frac{dx}{\sqrt{4+x^2}} \qquad 5)\ \int \frac{dt}{\sqrt{25t^2-9}}$$

$$3)\ \int \frac{dx}{4-x^2} \qquad 6)\ \int \frac{dy}{\sqrt{25+9y^2}}$$

Sol.-

1) let $z = \tan\theta \Rightarrow dz = \sec^2\theta \cdot d\theta$ $\qquad \tan\theta = \frac{z}{1}$

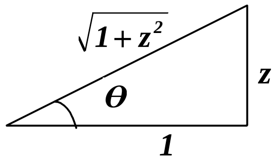
$$\int \frac{z^5\,dz}{\sqrt{1+z^2}} = \int \frac{\tan^5\theta \cdot \sec^2\theta\; d\theta}{\sqrt{1+\tan^2\theta}} = \int \tan^5\theta \cdot \sec\theta\; d\theta$$

$$= \int \tan\theta \cdot \sec\theta \left(\sec^2\theta - 1\right)^2 d\theta$$

$$= \int \sec^4\theta\left(\tan\theta \cdot \sec\theta\, d\theta\right) - 2\int \sec^2\theta\left(\tan\theta \cdot \sec\theta\, d\theta\right) + \int \tan\theta \cdot \sec\theta\, d\theta$$

$$= \frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta + c$$

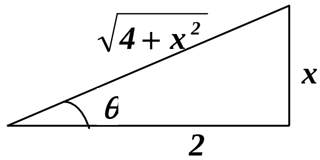
$$= \frac{1}{5}(\sqrt{1+z^2}\,)^5 - \frac{2}{3}(\sqrt{1+z^2}\,)^3 + \sqrt{1+z^2} + c$$



2) let $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta \cdot d\theta$ $\qquad \tan\theta = \frac{x}{2}$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\; d\theta}{\sqrt{4+4\tan^2\theta}} = \int \sec\theta\; d\theta = \ln\left|\sec\theta + \tan\theta\right| + c$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c$$

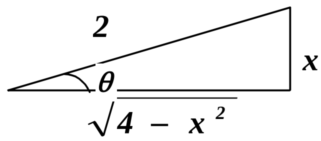


$$= \ln\left|\sqrt{4+x^2} + x\right| + c' \qquad where\;\; c' = c - \ln 2$$

3) let $x = 2\sin\theta \Rightarrow dx = 2\cos\theta \cdot d\theta$

$$\int \frac{dx}{4-x^2} = \int \frac{2\cos\theta\; d\theta}{4-4\sin^2\theta} = \frac{1}{2}\int \frac{d\theta}{\cos\theta} = \frac{1}{2}\int \sec\theta\, d\theta$$

$$= \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + c$$



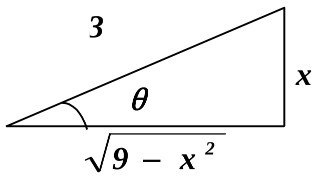
$$= \frac{1}{2}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{2+x}{\sqrt{(2-x)(2+x)}}\right| + c = \frac{1}{2}\ln\left|\sqrt{\frac{2+x}{2-x}}\right| + c = \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + c$$

4) let $x = 3\sin\theta \quad \Rightarrow \quad dx = 3\cos\theta \cdot d\theta$

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}\,3\cos\theta\,d\theta = 9\int \sin^2\theta\,d\theta$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + c$$



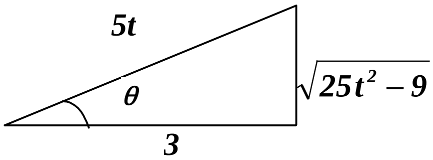
$$= \frac{9}{2}\left(\theta - \sin\theta\cdot\cos\theta\right) + c$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + c = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\cdot\sqrt{9-x^2} + c$$

5) let $5t = 3\sec\theta \quad \Rightarrow \quad 5dt = 3\sec\theta\cdot\tan\theta\,d\theta$

$$\int \frac{dt}{\sqrt{25t^2-9}} = \int \frac{\frac{3}{5}\sec\theta\cdot\tan\theta\,d\theta}{\sqrt{9\sec^2\theta-9}} = \frac{1}{5}\int \sec\theta\,d\theta$$

$$= \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + c$$



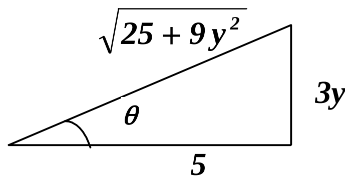
$$= \frac{1}{5}\ln\left|\frac{5t}{3} + \frac{\sqrt{25t^2-9}}{3}\right| + c$$

$$= \frac{1}{5}\ln\left|5t + \sqrt{25t^2-9}\right| + c' \quad \text{where } c' = c - \frac{1}{5}\ln 3$$

6) let $3y = 5\tan\theta \quad \Rightarrow \quad 3dy = 5\sec^2\theta\,d\theta$

$$\int \frac{dy}{\sqrt{25+9y^2}} = \int \frac{\frac{5}{3}\sec^2\theta\,d\theta}{\sqrt{25+25\tan^2\theta}} = \frac{1}{3}\int \sec\theta\,d\theta$$

$$= \frac{1}{3}\ln\left|\sec\theta + \tan\theta\right| + c$$



$$= \frac{1}{3}\ln\left|\frac{\sqrt{25+9y^2}}{5} + \frac{3y}{5}\right| + c$$

$$= \frac{1}{3}\ln\left|\sqrt{25+9y^2} + 3y\right| + c' \quad \text{where } c' = c - \frac{1}{3}\ln 5$$

EX-6 **Prove the following formulas:**

1) $\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c$ 2) $\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$

Proof.-

1) let $u = a\sin\theta \Rightarrow du = a\cos\theta \cdot d\theta$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \int \frac{a\cos\theta \cdot d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}\frac{u}{a} + c$$

2) let $u = a\tan\theta \Rightarrow du = a\sec^2\theta \cdot d\theta$

$$\int \frac{du}{a^2+u^2} = \int \frac{a\sec^2\theta \cdot d\theta}{a^2+a^2\tan^2\theta} = \frac{1}{a}\int d\theta = \frac{1}{a}\theta + c = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

6-4- Integral involving ax^2+bx+c :

By using the algebraic process called completing the square, we can convert any quadratic: ax^2+bx+c , $a \neq 0$ to the form: $a(u^2 \mp A^2)$ we can then use one of the trigonometric substitutions to write the expression as a times a single square term.

EX-7 **– Evaluate:**

1) $\int \frac{dx}{\sqrt{2x-x^2}}$

2) $\int \frac{dx}{2x^2+2x+1}$

3) $\int \frac{dx}{\sqrt{x^2+2x+2}}$

4) $\int \frac{dx}{\sqrt{1+x-x^2}}$

5) $\int \frac{dx}{\sqrt{x^2-2x-8}}$

Sol.

1) $\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{1-(x^2-2x+1)}} = \int \frac{dx}{\sqrt{1-(x-1)^2}}$

let $x-1 = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

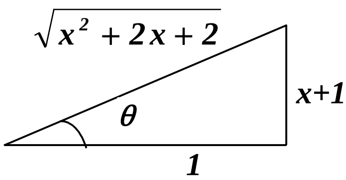
$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}(x-1) + c$$

$$2)\ \int \frac{dx}{2x^2+2x+1} = \frac{1}{2}\int \frac{dx}{x^2+x+1/2} = \frac{1}{2}\int \frac{dx}{\left(x+1/2\right)^2+1/4}$$

$$let \quad x+\frac{1}{2} = \frac{1}{2}\tan\theta \quad \Rightarrow \quad dx = \frac{1}{2}\sec^2\theta\, d\theta$$

$$\int \frac{dx}{2x^2+2x+1} = \frac{1}{2}\int \frac{1/2 \sec^2\theta\, d\theta}{1/4 \tan^2\theta + 1/4} = \int d\theta = \theta + c = \tan^{-1}(2x+1)+c$$

$$3)\ \int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{dx}{\sqrt{(x+1)^2+1}}$$



$$let \quad x+1 = \tan\theta \quad \Rightarrow \quad dx = \sec^2\theta\, d\theta$$

$$\int \frac{dx}{\sqrt{x^2+2x+2}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{\tan^2\theta+1}} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + c = \ln\left|\sqrt{x^2+2x+2} + x + 1\right| + c$$

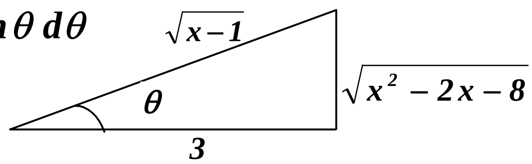
$$4)\ \int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{dx}{\sqrt{5/4 - \left(x-1/2\right)^2}}$$

$$let \quad x-\frac{1}{2} = \frac{\sqrt{5}}{2}\sin\theta \quad \Rightarrow \quad dx = \frac{\sqrt{5}}{2}\cos\theta\, d\theta$$

$$= \int \frac{\sqrt{5}/2 \ \cos\theta\, d\theta}{\sqrt{5/4 - 5/4 \sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$$

$$5)\ \int \frac{dx}{\sqrt{x^2-2x-8}} = \int \frac{dx}{\sqrt{(x-1)^2-9}}$$

$$let \quad x-1 = 3\sec\theta \quad \Rightarrow \quad dx = 3\sec\theta \cdot \tan\theta\, d\theta$$



$$= \int \frac{3\sec\theta \cdot \tan\theta\, d\theta}{\sqrt{9\sec^2\theta - 9}} = \int \sec\theta\, d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + c = \ln\left|\frac{x-1}{3} + \frac{\sqrt{x^2-2x-8}}{3}\right| + c$$

$$= \ln\left|x-1+\sqrt{x^2-2x-8}\right| + c' \qquad where \quad c' = c - \ln 3$$

6-5- Partial fractions:

Success in separating $\frac{f(x)}{g(x)}$ into a sum of partial fractions hinges on two things:-

1- **The degree of $f(x)$ must be less than the degree of $g(x)$. (If this is not case, we first perform a long division, and then work with the remainder term).**

2- **The factors of $g(x)$ must be known. If these two conditions are met we can carry out the following steps:**

Step I - let $x-r$ be a linear factor of $g(x)$. Suppose $(x-r)^m$ is the highest power of $(x-r)$ that divides $g(x)$. Then assign the sum of m partial factors to this factor, as follows:

$$\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\dots\dots\dots+\frac{A_m}{(x-r)^m}$$

Do this for each distinct linear factor of $f(x)$.

Step II - let x^2+px+q be an irreducible quadratic factor of $g(x)$. Suppose $(x^2+px+q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\dots\dots\dots+\frac{B_nx+C_n}{(x^2+px+q)^n}$$

Do this for each distinct linear factor of $g(x)$.

Step III - set the original fraction $\frac{f(x)}{g(x)}$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the sums in decreasing powers of x.

Step IV - equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EX-8 **– Evaluate the following integrals:**

1) $\int \frac{2x+5}{x^2-9}\,dx$

2) $\int \frac{x\,dx}{x^2+4x+3}$

3) $\int \frac{x^3-x}{(x^2+1)\cdot(x-1)^2}\,dx$

4) $\int \frac{\sin x\ dx}{\cos^2 x-5\cos x+4}$

5) $\int \frac{2x^2-3x+2}{(x-1)^2(x-2)}\,dx$

6) $\int \frac{x^3+4x^2}{x^2+4x+3}\,dx$

Sol.-

1) $\int \frac{2x+5}{x^2-9}\,dx = \int \frac{2x+5}{(x-3)\cdot(x+3)}\,dx$

$$\frac{2x+5}{(x-3)\cdot(x+3)} = \frac{A}{x-3}+\frac{B}{x+3} \quad \Rightarrow \quad 2x+5 = A(x+3)+B(x-3)$$

$$at \quad x=3 \quad \Rightarrow \quad 6A = 6+5 \quad \Rightarrow \quad A=\frac{11}{6}$$

$$at \quad x=-3 \quad \Rightarrow \quad -6B=-6+5 \quad \Rightarrow \quad B=\frac{1}{6}$$

$$\int \frac{2x+5}{x^2-9}\,dx = \int\left(\frac{11/6}{x-3}+\frac{1/6}{x+3}\right)dx = \frac{11}{6}\ln(x-3)+\frac{1}{6}\ln(x+3)+c$$

2) $\int \frac{x\,dx}{x^2+4x+3} = \int \frac{x\,dx}{(x+3)(x+1)}$

$$\frac{x}{(x+3)(x+1)} = \frac{A}{x+3}+\frac{B}{x+1} \quad \Rightarrow \quad x = A(x+1)+B(x+3)$$

$$at \quad x=-3 \quad \Rightarrow \quad A=\frac{3}{2} \quad and \quad at \quad x=-1 \quad \Rightarrow \quad B=-\frac{1}{2}$$

$$\int \frac{x\,dx}{x^2+4x+3} = \int\left(\frac{3/2}{x+3}+\frac{-1/2}{x+1}\right)dx = \frac{3}{2}\ln(x+3)-\frac{1}{2}\ln(x+1)+c$$

3) $\int \frac{x^3 - x}{(x^2+1)(x-1)^2}\,dx = \int \frac{x(x-1)(x+1)}{(x^2+1)(x-1)^2}\,dx = \int \frac{x^2+x}{(x^2+1)(x-1)}\,dx$

$$\frac{x^2+x}{(x^2+1)(x-1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} \quad \Rightarrow \quad x^2 + x = (Ax+B)(x-1) + C(x^2+1)$$

$$x^2 + x = (A+C)x^2 + (-A+B)x + (-B+C)$$

$$\left.\begin{array}{l} A+C=1 \;\; ......(1) \\ -A+B=1 \;\; ......(2) \\ -B+C=0 \;\; ......(3) \end{array}\right\} \Rightarrow \quad A=0 \;\;,\;\; B=1 \;,\; C=1$$

$$\int \frac{x^3-x}{(x^2+1)(x-1)^2}\,dx = \int \left(\frac{1}{x^2+1} + \frac{1}{x-1} \right) dx = \tan^{-1} x + \ln(x-1) + c$$

4) let $y = cosx \quad \Rightarrow \quad dy = -sinx\,dx$

$$\int \frac{\sin x\;dx}{\cos^2 x - 5\cos x + 4} = -\int \frac{dy}{y^2 - 5y + 4} = -\int \frac{dy}{(y-4)(y-1)}$$

$$\frac{dy}{(y-4)(y-1)} = \frac{A}{y-4} + \frac{B}{y-1} \quad \Rightarrow \quad 1 = A(y-1) + B(y-4)$$

at $y=4 \;\Rightarrow\; A = \frac{1}{3}$ and at $y = 1 \;\Rightarrow\; B = -\frac{1}{3}$

$$\int \frac{\sin x\;dx}{\cos^2 x - 5\cos x + 4} = -\int \left(\frac{1/3}{y-4} + \frac{-1/3}{y-1} \right) dy$$

$$= -\frac{1}{3}\ln(y-4) + \frac{1}{3}\ln(y-1) + c = -\frac{1}{3}\ln(\cos x - 4) + \frac{1}{3}\ln(\cos x - 1) + c$$

5) $\frac{2x^2 - 3x + 2}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$

$$2x^2 - 3x + 2 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

$$\left.\begin{array}{l} A+C=2 \;\; ....(1) \\ -3A+B-2C=-3 \;......(2) \\ 2A-2B+C=2 \;...........(3) \end{array}\right\} \Rightarrow \quad A=-2 \;\;,\;\; B=-1 \;,\; C=4$$

$$\int \frac{2x^2-3x+2}{(x-1)^2(x-2)}\,dx = \int \left(\frac{-2}{x-1} + \frac{1}{(x-1)^2} + \frac{4}{x-2} \right) dx$$

$$= -2\ln(x-1) + \frac{1}{x-1} + 4\ln(x-2) + C$$

$$6)\ \frac{x^3+4x^2}{x^2+4x+3}=x-\frac{3x}{(x+3)(x+1)}$$

$$\begin{array}{r|l} & x \\ x^2+4x+3 & x^3+4x^2 \\ & \mp x^3\mp 4x^2\mp 3x \\ \hline & -3x \end{array}$$

$$\frac{3x}{(x+3)(x+1)}=\frac{A}{x+3}+\frac{B}{x+1}\quad\Rightarrow\quad 3x=A(x+1)+B(x+3)$$

$$at\quad x=-3\ \Rightarrow A=\frac{9}{2}\quad and\quad at\quad x=-1\ \Rightarrow B=-\frac{3}{2}$$

$$\int\frac{x^3+4x^2}{x^2+4x+3}dx=\int\left(x-\frac{9/2}{x+3}+\frac{3/2}{x+1}\right)dx$$

$$=\frac{x^2}{2}-\frac{9}{2}ln(x+3)-\frac{3}{2}ln(x+1)+c$$

6-6- Rational functions of *sinx* and *cosx*, and other trigonometric integrals:

We assume that $z=tan\frac{x}{2}$ **then** $x=2\,tan^{-1}z$ **and** $dx=\frac{2}{1+z^2}dz$

Since

$$cos^2\frac{x}{2}=\frac{1+cos\,x}{2}\quad\Rightarrow\quad cosx=2\,cos^2\frac{x}{2}-1=\frac{2}{sec^2\frac{x}{2}}-1$$

$$=\frac{2}{tan^2\frac{x}{2}+1}-1=\frac{2}{z^2+1}-1\quad\Rightarrow\quad cosx=\frac{1-z^2}{1+z^2}$$

Since

$$sin\,x=2\,sin\frac{x}{2}\cdot cos\frac{x}{2}=\frac{2sin\frac{x}{2}}{cos\frac{x}{2}}\cdot cos^2\frac{x}{2}=2\,tan\frac{x}{2}\cdot\frac{1}{sec^2\frac{x}{2}}$$

$$=2\,tan\frac{x}{2}\cdot\frac{1}{tan^2\frac{x}{2}+1}\quad\Rightarrow\quad sinx=\frac{2z}{1+z^2}$$

EX-9 **– Evaluate:**

1) $\int \frac{dx}{1+\sin x+\cos x}$

2) $\int \frac{dx}{\sin x+\tan x}$

3) $\int \frac{dx}{2+\sin x}$

4) $\int \frac{3\ dx}{2+4\sin x}$

5) $\int \sec x\, dx$

6) $\int \frac{\cos x\, dx}{1-\cos x}$

Sol.-

1) $$\int \frac{dx}{1+\sin x+\cos x}=\int \frac{\frac{2}{1+z^2}dz}{1+\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}}=\int \frac{dz}{1+z}$$

$$=\ln\left|1+z\right|+c=\ln\left|1+\tan\frac{x}{2}\right|+c$$

2) $$\int \frac{dx}{\sin x+\tan x}=\int \frac{\frac{2}{1+z^2}dz}{\frac{2z}{1+z^2}+\frac{2z}{1-z^2}}=\frac{1}{2}\int\left(\frac{1}{z}-z\right)dz$$

$$=\frac{1}{2}\left[\ln z-\frac{z^2}{2}\right]+c=\frac{1}{2}\left[\ln\tan\frac{x}{2}-\frac{1}{2}\tan^2\frac{x}{2}\right]+c$$

3) $$\int \frac{dx}{2+\sin x}=\int \frac{\frac{2}{1+z^2}dz}{2+\frac{2z}{1+z^2}}=\int \frac{dz}{z^2+z+1}=\int \frac{dz}{(z+\frac{1}{2})^2+\frac{3}{4}}$$

$$let\quad z+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta\ \Rightarrow\ dz=\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$$

$$\int \frac{dx}{2+\sin x}=\int \frac{\frac{\sqrt{3}}{2}\sec^2\theta\, d\theta}{\frac{3}{4}\tan^2\theta+\frac{3}{4}}=\frac{2}{\sqrt{3}}\int d\theta=\frac{2}{\sqrt{3}}\theta+c$$

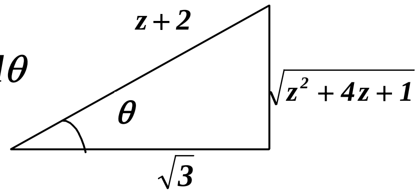
$$=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)+c=\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}+1}{\sqrt{3}}\right)+c$$

$$4)\ \int \frac{3\ dx}{2+4\sin x} = \frac{3}{2}\int \frac{dx}{1+2\sin x} = \frac{3}{2}\int \frac{\frac{2}{z^2+1}dz}{1+2\frac{2z}{z^2+1}} = 3\int \frac{dz}{z^2+4z+1}$$

$$= 3\int \frac{dz}{(z+2)^2-3} = \int \frac{dz}{(\frac{z+2}{\sqrt{3}})^2-1}$$

$$let \quad \frac{z+2}{\sqrt{3}} = \sec\theta \quad \Rightarrow \quad dz = \sqrt{3}\sec\theta\cdot\tan\theta\, d\theta$$

$$\int \frac{3\ dx}{2+4\sin x} = \int \frac{\sqrt{3}\sec\theta\cdot\tan\theta\, d\theta}{\sec^2\theta-1} = \sqrt{3}\int \frac{\sec\theta}{\tan\theta}d\theta$$



$$= \sqrt{3}\int \csc\theta\, d\theta = -\sqrt{3}\ln\left|\csc\theta+\cot\theta\right|+c$$

$$= -\sqrt{3}\ln\left|\frac{z+2}{\sqrt{z^2+4z+1}}+\frac{\sqrt{3}}{\sqrt{z^2+4z+1}}\right|+c = -\sqrt{3}\ln\left|\frac{\tan\frac{x}{2}+2+\sqrt{3}}{\sqrt{\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1}}\right|+c$$

$$5)\ \int \sec x\, dx = \int \frac{1+z^2}{1-z^2}\cdot\frac{2}{1+z^2}dz = 2\int \frac{1}{(1-z)(1+z)}dz$$

$$\frac{1}{(1-z)(1+z)} = \frac{A}{1-z}+\frac{B}{1+z} \quad \Rightarrow \quad A(1+z)+B(1-z)=1$$

$$at \quad z=1 \quad \Rightarrow \quad A=\frac{1}{2} \quad and \quad at \quad z=-1 \quad \Rightarrow \quad B=\frac{1}{2}$$

$$\int \sec x\, dx = 2\int\left(\frac{1/2}{1-z}+\frac{1/2}{1+z}\right)dz = -\ln(1-z)+\ln(1+z)+c$$

$$= \ln\left(1+\tan\frac{x}{2}\right)-\ln\left(1-\tan\frac{x}{2}\right)+c = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|+c$$

$$By\ substituting \quad \tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}} \quad implies$$

$$\int \sec x\, dx = \ln\left|\sec x+\tan x\right|+c$$

$$6)\ \int \frac{\cos x\, dx}{1-\cos x} = \int \frac{\frac{1-z^2}{1+z^2}}{1-\frac{1-z^2}{1+z^2}} \cdot \frac{2}{1+z^2}\, dz = \int \frac{1-z^2}{(1+z^2)\, z^2}\, dz$$

$$\frac{1-z^2}{(1+z^2)\, z^2} = \frac{A}{z} + \frac{B}{z^2} + \frac{C\, z + D}{1+z^2} \qquad \Rightarrow$$

$$Az + Az^3 + B + Bz^2 + Cz^3 + Dz^2 = 1 - z^2$$

$$\left.\begin{array}{l} A + C = 0 \ \ ......(1) \\ B + D = -1 \ \ ....(2) \\ A = 0 \ \ ....(3) \\ B = 1 \ \ ....(4) \end{array}\right\} \Rightarrow \quad A = 0 \ \ ,\ \ B = 1\ ,\ C = 0\ ,\ D = -2$$

$$\int \frac{\cos x\, dx}{1-\cos x} = \int \left(\frac{1}{z^2} - \frac{2}{z^2+1} \right) dz = -\frac{1}{z} - 2\tan^{-1} z + c$$

$$= -\frac{1}{\tan\frac{x}{2}} - 2 \cdot \frac{x}{2} + c = -\cot\frac{x}{2} - x + c$$

Problems – 6

Evaluate the following integrals:

1) $\int \frac{x^3}{x-1}\,dx$ (ans.: $\frac{1}{3}x^3 - \frac{1}{2}x^2 + x + \ln(x-1) + c$)

2) $\int \frac{3x+2}{3x-1}\,dx$ (ans.: $x + \ln(3x-1) + c$)

3) $\int x^2 \cdot e^{-x}\,dx$ (ans.: $-e^{-x}(x^2+2x+2)+c$)

4) $\int x \cdot \sin x^2\,dx$ (ans.: $-\frac{1}{2}\cos x^2 + c$)

5) $\int \sqrt{x^2-1}\,dx$ (ans.: $\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right| + c$)

6) $\int \frac{3x+13}{(5x-1)(7x+2)}$ (ans.: $\frac{4}{5}\ln|5x-1| - \frac{5}{7}\ln|7x+2| + c$)

7) $\int \frac{2x-3}{(x-1)(x-2)(x+3)}\,dx$ (ans.: $\frac{1}{4}\ln|x-1| + \frac{1}{5}\ln|x-2| - \frac{9}{20}\ln|x+3| + c$)

8) $\int \frac{dx}{x^4-1}$ (ans.: $\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1} x + c$)

9) $\int \ln x\,dx$ (ans.: $x \cdot \ln x - x + c$)

10) $\int \tan^{-1} x\,dx$ (ans.: $x \cdot \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$)

11) $\int x \cdot \ln x\,dx$ (ans.: $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$)

12) $\int x \cdot \tan^{-1} x\,dx$ (ans.: $\frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + c$)

13) $\int x^2 \cdot \cos ax\,dx$ (ans.: $\frac{x^2}{a}\sin ax + \frac{2x}{a^2}\cos ax - \frac{2}{a^3}\sin ax + c$)

14) $\int \sin(\ln x)\,dx$ (ans.: $\frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + c$)

15) $\int \ln(a^2+x^2)\,dx$ (ans.: $x \cdot \ln(a^2+x^2) - 2x + 2a\tan^{-1}\frac{x}{a} + c$)

$16)\ \int x \cdot \sin^{-1} x\, dx$ $\quad (ans.:\ \frac{x^2}{2}\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{x}{4}\sqrt{1-x^2} + c\,)$

$17)\ \int \cos^4 x\ dx$ $\quad (ans.:\ \frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c\,)$

$18)\ \int \cos^{\frac{2}{3}} x \cdot \sin^5 x\, dx$ $\quad (ans.:\ -\frac{3}{5}\cos^{\frac{5}{3}} x + \frac{6}{11}\cos^{\frac{11}{3}} x - \frac{3}{17}\cos^{\frac{17}{3}} x + c\,)$

$19)\ \int x \cdot \sin x\, dx$ $\quad (ans.:\ -x \cdot \cos x + \sin x + c\,)$

$20)\ \int x^2\sqrt{1-x}\, dx$ $\quad (ans.:\ -\frac{2}{105}\sqrt{(1-x)^3}\left(15x^2 + 12x + 8\right) + c\,)$

$21)\ \int \sin^2 x \cdot \cos^2 x\ dx$ $\quad (ans.:\ \frac{1}{32}\left(4x - \sin 4x\right) + c\,)$

$22)\ \int \sec^3 x \cdot \tan^2 x\ dx$ $\quad (ans.:\ \frac{1}{4}\sec^3 x \cdot \tan x - \frac{1}{8}\sec x \cdot \tan x$

$- \frac{1}{8}\ln\left|\sec x + \tan x\right| + c\,)$

$23)\ \int x\left(\cos^3 x^2 - \sin^3 x^2\right)dx$ $\quad (ans.:\ \frac{1}{2}\sin x^2 - \frac{1}{6}\sin^3 x^2 + \frac{1}{2}\cos x^2$

$- \frac{1}{6}\cos^3 x^2 + c\,)$

$24)\ \int \frac{dx}{\sqrt{x}\sqrt{1-x}}$ $\quad (ans.:\ 2\sin^{-1}\sqrt{x} + c\,)$

$25)\ \int \frac{dx}{\sqrt{x}\cdot(1+\sqrt{x}\,)}$ $\quad (ans.:\ 2\ln(1+\sqrt{x}\,) + c\,)$

$26)\ \int \frac{dx}{x\sqrt{2 - 3\ln^2\sqrt{x}}}$ $\quad (ans.:\ \frac{2}{\sqrt{3}}\sin^{-1}(\frac{\sqrt{3}}{2}\ln x) + c\,)$

$27)\ \int \frac{e^{2x}\, dx}{\sqrt[3]{1+e^x}}$ $\quad (ans.:\ \frac{3}{2}\cdot e^x \cdot \sqrt[3]{(1+e^x)^2} - \frac{9}{10}\sqrt[3]{(1+e^x)^5} + c\,)$

$28)\ \int \frac{dy}{y(2y^3+1)^2}$ $\quad (ans.:\ \frac{1}{3}\ln(\frac{2y^3}{2y^3+1}) - \frac{2y^3}{3(2y^3+1)} + c\,)$

$29)\ \int \frac{x\, dx}{1+\sqrt{x}}$ $\quad (ans.:\ \frac{2}{3}\sqrt{x^3} - x + 2\sqrt{x} - 2\ln(\sqrt{x}+1) + c\,)$

$30)\ \int \frac{dt}{e^t - 1}$ $\quad (ans.:\ \ln(e^t - 1) - t + c\,)$

31) $\int \frac{d\theta}{1-\tan^2\theta}$ $\quad$ (ans.: $\frac{1}{2}\theta + \frac{1}{4}\ln\left|\sec 2\theta + \tan 2\theta\right| + c$)

32) $\int e^x \cdot \cos 2x \; dx$ $\quad$ (ans.: $\frac{e^x}{5}\cos 2x + \frac{2}{5}e^x \sin 2x + c$)

33) $\int \frac{\cot\theta \, d\theta}{1+\sin^2\theta}$ $\quad$ (ans.: $\ln\frac{\sin\theta}{\sqrt{1+\sin^2\theta}} + c$)

34) $\int \frac{e^{4t}}{(1+e^{2t})^{\frac{2}{3}}} dt$ $\quad$ (ans.: $\frac{3}{2}e^{2t}(1+e^{2t})^{\frac{1}{3}} - \frac{9}{8}(1+e^{2t})^{\frac{3}{4}} + c$)

35) $\int \frac{x^3+x^2}{x^2+x-2} dx$ $\quad$ (ans.: $\frac{x^2}{2} + \frac{4}{3}\ln(x+2) + \frac{2}{3}\ln(x-1) + c$)

36) $\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}} \; dx$ $\quad$ (ans.: $\frac{1}{3}(2\sqrt{3e^{2x}-6e^x-1} +$

$\sqrt{3}\ln\left|\sqrt{3}(e^x-1) + \sqrt{3e^{2x}-6e^x-1}\right| + c$)

37) $\int \frac{dy}{(2y+1)\sqrt{y^2+y}}$ $\quad$ (ans.: $\sec^{-1}(2y+1) + c$)

38) $\int (1-x^2)^{\frac{3}{2}} \; dx$ $\quad$ (ans.: $\frac{e^x}{5}\cos 2x + \frac{2}{5}e^x \sin 2x + c$)

39) $\int \frac{\tan^{-1} x}{x^2} dx$ $\quad$ (ans.: $\ln\frac{x}{\sqrt{x^2+1}} - \frac{\tan^{-1} x}{x^2} + c$)

40) $\int x \cdot \sin^2 x \; dx$ $\quad$ (ans.: $\frac{x^2}{4} - \frac{x}{4}\sin 2x + \frac{1}{8}\cos 2x + c$)

41) $\int \frac{dt}{t^4+4t^2+3}$ $\quad$ (ans.: $\frac{1}{2}\tan^{-1} t - \frac{1}{2\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + c$)

42) $\int \frac{8dx}{x^4+2x^3}$ $\quad$ (ans.: $\ln\frac{x}{x+2} + \frac{2}{x} - \frac{2}{x^2} + c$)

43) $\int \frac{\cos x \; dx}{\sqrt{1+\cos x}}$ $\quad$ (ans.: $\sqrt{2}(2\sin\frac{x}{2} - \ln\left|\sec\frac{x}{2} + \tan\frac{x}{2}\right|) + c$)

44) $\int \frac{x \; dx}{x+\sqrt{x}+1}$ $\quad$ (ans.: $x - 2\sqrt{x} + \frac{4}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{x}+1}{\sqrt{3}} + c$)

45) $\int \frac{dt}{\sec^2 t + \tan^2 t}$ $\quad$ (ans.: $\sqrt{2}\tan^{-1}(\sqrt{2}\tan t) - t + c$)

$46)\ \int \frac{dx}{1+\cos^2 x}$ $\qquad (ans.:\ \frac{1}{\sqrt{2}}\tan^{-1}(\frac{1}{\sqrt{2}}\tan x)+c)$

$47)\ \int \ln(\sqrt{x}+\sqrt{1+x})\,dx$ $\qquad (ans.:\ x\ln(\sqrt{x}+\sqrt{1+x})-\frac{\sqrt{x^2+x}}{2}$

$+\frac{1}{4}\ln\left|2x+1+2\sqrt{x^2+x}\right|+c)$

$48)\ \int x\ln(x^3+x)\,dx$ $\qquad (ans.:\ \frac{x^2}{2}\ln(x^3+x)-\frac{3}{4}x^2+\frac{1}{2}\ln(x^2+1)+c)$

$49)\ \int \frac{\cos x\,dx}{\sqrt{4-\cos^2 x}}$ $\qquad (ans.:\ \ln\left|\sqrt{3+\sin^2 x}+\sin x\right|+c)$

$50)\ \int \frac{\sec^2 x\,dx}{\sqrt{4-\sec^2 x}}$ $\qquad (ans.:\ \sin^{-1}(\frac{1}{\sqrt{3}}\tan x)+c)$

$51)\ \int \frac{dt}{t-\sqrt{1-t^2}}$ $\qquad (ans.:\ \frac{1}{2}\ln(t-\sqrt{1-t^2})-\frac{1}{2}\sin^{-1}t+c)$

$52)\ \int e^{-x}\cdot\tan^{-1}e^{x}\,dx$ $\qquad (ans.:\ -e^{-x}\cdot\tan^{-1}e^{x}+x-\frac{1}{2}\ln(1+e^{2x})+c)$

$53)\ \int \sin^{-1}\sqrt{x}\,dx$ $\qquad (ans.:\ x\sin^{-1}\sqrt{x}-\frac{1}{2}\sin^{-1}\sqrt{x}+\frac{1}{2}\sqrt{x-x^2}+c)$

$54)\ \int \frac{\cos 2x-1}{\cos 2x+1}\,dx$ $\qquad (ans.:\ x-\tan x+c)$

Chapter seven

Application of integrals

7-1- <u>Definite integrals</u>:

If $f(x)$ is continuous in the interval $a \le x \le b$ and it is integrable in the interval then the area under the curve:-

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is any function such that $F'(x) = f(x)$ in the interval.

Some of the more useful properties of the definite integral are:-

1) $\int_a^b c\, f(x)\,dx = c\int_a^b f(x)\,dx$, *where c is constant.*

2) $\int_a^b \left(f(x) \mp g(x)\right)dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx$

3) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4) *Let* $a < c < b$ *then* $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

5) $\int_a^a f(x)\,dx = 0$

6) *If* $f(x) \ge 0$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \ge 0$

7) *If* $f(x) \le g(x)$ *for* $a \le x \le b$ *then* $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$